function.[8] These phase functions can be obtained by reflecting zeros about the unit circle to their conjugate reciprocal locations by multiplying $H(z)$ by $[\frac{1-a^*z}{1-az^{-1}}]^{\pm 1}$. Because the all-pass function with unity power $(+1)$ has negative phase (Exercise 2.10), if we flip a zero of a minimum-phase function outside the unit circle, then the resulting phase is more negative than the original. The term *minimum phase-lag* would thus be more precise than the commonly used *minimum phase* [7].

There are a number of important properties of minimum-phase sequences that have important consequences for speech modeling and processing. A minimum-phase sequence is uniquely specified by the magnitude of its Fourier transform. This result will be proven formally in Chapter 10 in the context of the complex cepstrum representation of a sequence, but can be seen intuitively from the example of a stable rational z-transform. In this case, for a given Fourier-transform magnitude, there is only one sequence with all zeros (or poles) inside the unit circle. Likewise, the Fourier transform phase of a minimum-phase sequence $H(\omega)$ uniquely specifies the sequence (to within a scale factor).

Another useful property relates to the energy concentration of a minimum-phase sequence. From Parseval's theorem, all sequences with the same Fourier transform magnitude have the same energy, but when we flip zeros (or poles) inside and outside the unit circle to their conjugate reciprocal locations, this energy gets distributed along the time axis in different ways. It can be shown that a finite-length (all-zero) minimum-phase sequence has energy most concentrated near (and to the right of) the time origin, relative to all other finite-length causal sequences with the same Fourier transform magnitude, and thus tends to be characterized by an abrupt onset or what is sometimes referred to as a fast "attack" of the sequence.[9] This property can be formally expressed as [9]

$$\sum_{n=0}^{m} |h_{rmp}[n]|^2 \geq \sum_{n=0}^{m} |h[n]|^2, \quad m \geq 0 \tag{2.23}$$

where $h[n]$ is a causal sequence with the Fourier transform magnitude equal to that of the reference minimum-phase sequence $h_{rmp}[n]$. As zeros are flipped outside the unit circle, the energy of the sequence is delayed in time, the maximum-phase counterpart having maximum energy delay (or phase lag) [9]. Similar energy localization properties are found with respect to poles. However, because causality strictly cannot be made to hold when a z-transform contains maximum-phase poles, it is more useful to investigate how the energy of the sequence shifts with respect to the time origin. As illustrated in Example 2.11, flipping poles from inside to outside the unit circle to their conjugate reciprocal location moves energy to the left of the time origin, transforming the fast attack of the minimum-phase sequence to a more gradual onset. We will see throughout the text that numerous speech analysis schemes result in a minimum-phase vocal tract impulse response estimate. Because the vocal tract is not necessarily minimum phase, synthesized speech may be characterized in these cases by an unnaturally abrupt vocal tract impulse response.

[8] Because we assume causality and stability, the poles lie inside the unit circle. Different phase functions, for a specified magnitude, therefore are not contributed by the poles.

[9] It is of interest to note that a sufficient but not necessary condition for a causal sequence to be minimum phase is that $|h[0]| > \sum_{n=1}^{\infty} |h[n]|$ [9].