and therefore

$$2X(\omega) = S_q(\omega) + S_q^*(-\omega).$$

(b) Let $S_q(\omega)$ and $S_a(\omega)$ denote the Fourier transforms of the quadrature and analytic signals, respectively. Show that

$$\begin{aligned} 2S_a(\omega) &= 0, \quad \omega < 0 \\ &= S_q(\omega) + S_q^*(-\omega), \quad \omega \geq 0. \end{aligned}$$

(c) Using the following mean-squared difference as a criterion of closeness between the quadrature and analytic signals:

$$C = \sum_{n=-\infty}^{\infty} |2s_a[n] - s_q[n]|^2$$

(the factor of 2 is needed to account for removing negative frequencies in obtaining the analytic signal), and using Parseval's Theorem and your result from part (b), show that

$$C = \frac{2}{\pi} \int_{-\pi}^{0} |S_q(\omega)|^2 d\omega$$

which is twice the energy of the quadrature signal for negative frequencies.

(d) Based on the closeness criterion of part (c), state whether the analytic and quadrature representations are equivalent for each of the following two signals:

$$\begin{aligned} x_1[n] &= \cos(\omega n) \\ x_2[n] &= a^n \cos(\omega n) \end{aligned}$$

where $0 \leq \omega < \pi$. Explain your reasoning.

2.8 Consider a complex continuous-time signal of the form $s(t) = a(t)e^{j\phi(t)}$. Show that the bandwidth of $s(t)$, as defined in its discrete-time counterpart of Equation (2.4), can be expressed as

$$B = \int_{-\infty}^{\infty} \left(\frac{da(t)}{dt}\right)^2 dt + \int_{-\infty}^{\infty} \left(\frac{d\phi(t)}{dt} - \bar{\Omega}\right)^2 a^2(t)dt \tag{2.36}$$

where $\bar{\Omega}$ is the signal's average frequency, i.e., $\bar{\Omega} = \int_{-\infty}^{\infty} \Omega|S(\Omega)|^2 d\Omega$. *Hint:* Use the relation $\int_{-\infty}^{\infty} \Omega^2 |S(\Omega)|^2 d\Omega = \int_{-\infty}^{\infty} |\frac{d}{dt}s(t)|^2 dt$.

2.9 The autocorrelation function for a real-valued stable sequence $x[n]$ is defined as

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k].$$

(a) Show that the z-transform of $c_{xx}[n]$ is

$$C_{xx}(z) = X(z)X(z^{-1}).$$

Determine the region of convergence for $C_{xx}(z)$.

(b) Suppose that $x[n] = a^n u[n]$. Sketch the pole-zero plot for $C_{xx}(z)$, including the region of convergence. Also find $c_{xx}[n]$ by evaluating the inverse z-transform of $C_{xx}(z)$.