which is the Fourier transform of $h[n, m]$ at time n evaluated with respect to the variable m and referred to as the *time-varying frequency response*. Equivalently, we can write the time-varying frequency response in terms of Green's function as (Exercise 2.17)

$$H(n, \omega) = e^{j\omega n} \sum_{m=-\infty}^{\infty} g[n, m] e^{-j\omega m} \tag{2.30}$$

which, except for the linear phase factor $e^{j\omega n}$, is the Fourier transform of Green's function at time n.

Because the system of interest is linear, its output to an arbitrary input $x[n]$ is given by the following superposition [10] (Exercise 2.15):

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[n, m] x[n - m] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(n, \omega) X(\omega) e^{j\omega n} d\omega \end{aligned} \tag{2.31}$$

so that the output $y[n]$ of $h[n, m]$ at time n is the inverse Fourier transform of the *product* of $X(\omega)$ and $H(n, \omega)$, $X(\omega)H(n, \omega)$ which can be thought of as a generalization of the Convolution Theorem for linear time-invariant systems. This generalization, however, can be taken only so far. For example the elements of a cascade of two time-varying linear systems, i.e., $H_1(n, \omega)$ followed by $H_2(n, \omega)$, do not generally combine in the frequency domain by multiplication and the elements cannot generally be interchanged, as illustrated in the following example. Consequently, care must be taken interchanging the order of time-varying systems in the context of speech modeling and processing.

EXAMPLE 2.12 Consider the linear time-varying multiplier operation

$$y[n] = x[n] e^{j\omega_o n}$$

cascaded with a linear time-invariant ideal low-pass filter $h[n]$, as illustrated in Figure 2.10. Then, in general, $(x[n]e^{j\omega_o n}) * h[n] \neq (x[n] * h[n])e^{j\omega_o n}$. For example, let $x[n] = e^{j\frac{\pi}{3}n}$, $\omega_o = \frac{\pi}{3}$, and $h[n]$ have lowpass cutoff frequency at $\frac{\pi}{2}$. When the lowpass filter follows the multiplier, the output is zero; when the order is interchanged, the output is nonzero. ▲

We will see in following chapters that under certain "slowly varying" conditions, linear time-varying systems can be approximated by linear time-invariant systems. The accuracy of this approximation will depend on the time-duration over which we view the system and its input, as well as the rate at which the system changes. More formal conditions have been derived by Matz and Hlawatsch [6] under which a "transfer function calculus" is allowed for time-varying systems.