$|x[n]|^2$ and $|X(\omega)|^2$ viewed as energy densities, we will see later in the text, are analogous to the probability density function used in defining the variance of a random variable [2]. It follows that normalizing the magnitude functions in Equation (2.4) by the total signal energy ensures probability density-like functions that integrate to unity.

The uncertainty principle states that the product of signal duration and bandwidth cannot be less than a fixed limit. In continuous time, this condition is given by

$$D(x)B(x) \geq \frac{1}{2}. \tag{2.5}$$

Proof of the uncertainty principle is given by first applying Parseval's Theorem to obtain

$$D(x)B(x) \geq \left| \int_{-\pi}^{\pi} \Omega X^*(\Omega) \frac{dX(\Omega)}{d\Omega} d\Omega \right|^2 \tag{2.6}$$

from which Equation (2.5) follows. The reader is led through the derivation in Exercise 2.5. Bounding relations also can be found for discrete time (11], [12]. The principle implies that a wide signal gives a narrow Fourier transform, and a narrow signal gives a wide Fourier transform.[5] The uncertainty principle will play a major role in spectrographic, and, more generally, time-frequency analysis of speech, especially when the speech waveform consists of dynamically changing events or closely-spaced time or frequency components.

It is important to look more carefully at our definition of bandwidth in Equation (2.4). Observe that for a real sequence, from the conjugate-symmetry property, the Fourier transform magnitude is even. Thus the average frequency is zero and the bandwidth is determined by the distance between the spectral energy concentrations in positive and negative frequency. The resulting bandwidth, therefore, is not necessarily indicative of the distribution of energy of physically meaningful quantities such as system resonances (Exercise 2.4). The bandwidth of the signals with discrete-time Fourier transform magnitudes in Figure 2.2c is such a case. As a consequence, complex sequences, as those corresponding to the transform magnitudes in Figure 2.2a, or only the positive frequencies of a real sequence, are used in computing bandwidth. In the latter case, we form a sequence $s[n]$ with Fourier transform

$$\begin{aligned} S(\omega) &= X(\omega), \quad 0 \leq \omega \leq \pi \\ &= 0, \qquad\quad -\pi < \omega < 0 \end{aligned}$$

where we implicitly assume that $S(\omega)$ is periodic with period 2π. The sequence $s[n]$ must be complex because conjugate symmetry is not satisfied, and so can be represented by real and

[5] The nomenclature "uncertainty" is somewhat misleading because there is no uncertainty in the measurement of the signal or its Fourier transform. This terminology evolved from Heisenberg's uncertainty principle in the probabilistic context of quantum mechanics where it was discovered that the position and the momentum of a particle cannot be measured at a particular time simultaneously with absolute certainty. There is a mathematical similarity, and *not* a similarity in the physical interpretation, in Heisenberg's uncertainty principle with Equation (2.5) because the position and momentum functions are related through the Fourier transform. The squared magnitude of the Heisenberg functions represent the *probability* of measuring a particle with a certain position and momentum, respectively, unlike the deterministic magnitude of a signal and its Fourier transform [2].