**2.5** In this exercise, you are asked to prove the uncertainty principle in Equation (2.5).

**(a)** Show that if $x(t)$ has Fourier transform $X(\Omega)$, then differentiating in frequency results in the pair

$$tx(t) \leftrightarrow j\frac{d}{d\Omega}X(\Omega).$$

**(b)** Using your result from part (a), derive Equation (2.6) using the Schwartz inequality: $|\int_a^b x(u)y(u)du|^2 \leq \int_a^b |x(u)|^2 du \int_a^b |y(u)|^2 du$.

**(c)** Complete the derivation of the uncertainty principle. *Hint:* Write the Fourier transform $X(\Omega)$ in terms of its magnitude and phase, integrate, and then determine terms that are always positive.

**2.6** In this exercise, you are asked to investigate the analytic signal representation.

**(a)** Show that the real part of the analytic signal representation of $x[n]$ in Equation (2.7) is given by $s_r[n] = x[n]/2$. Show that the imaginary part $s_i[n]$ is obtained by applying the Hilbert transformer

$$H(\omega) = \begin{cases} -j & 0 \leq \omega < \pi \\ +j & -\pi \leq \omega < 0 \end{cases}$$

to $s_r[n] = x[n]/2$ as in Equation (2.8).

**(b)** Find the output of the Hilbert transformer when the input sequence is given by

$$x[n] = \cos(\omega_0 n)$$

where $0 < \omega_0 < \pi$.

**(c)** Find the magnitude and phase of the analytic signal

$$s[n] = s_r[n] + js_i[n]$$

for the sequence $x[n]$ of part (b).

**(d)** Repeat parts (b) and (c) for the input sequence

$$x[n] = \cos(\omega_0 n)\left[\frac{\sin(\omega_1 n)}{\pi n}\right]$$

where $0 < \omega_0 - \omega_1 < \pi$ and $0 < \omega_0 + \omega_1 < \pi$.

**2.7** In this exercise, you are asked to find the condition under which the analytic and quadrature representations of the real signal $x[n] = a[n]\cos(\phi[n])$ are equivalent. We write the quadrature and analytic signal representations, respectively, as

$$s_q[n] = a[n]e^{j\phi[n]}$$

and

$$s_a[n] = \frac{1}{2\pi}\int_0^{\pi} X(\omega)e^{j\omega n}d\omega.$$

**(a)** Show that

$$S_q(\omega) = X(\omega) + j\sum_{n=-\infty}^{\infty} a[n]\sin(\phi[n])e^{-j\omega n}$$