Using the linearity of the Fourier transform, we can generalize the previous result to a sinusoidal sequence as well as to multiple complex exponentials and sines. Figure 2.3b–d illustrates the Fourier transforms of the following three classes of sequences:

**Sinusoidal sequence**

$$A\cos(\omega_o n + \phi) \leftrightarrow \pi A e^{j\phi}\delta(\omega - \omega_o) + \pi A e^{-j\phi}\delta(\omega + \omega_o)$$

**Multiple complex exponentials**

$$\sum_{k=0}^{N} a_k e^{j\omega_k n + \phi_k} \leftrightarrow \sum_{k=0}^{N} 2\pi a_k e^{j\phi_k}\delta(\omega - \omega_k)$$

**Multiple sinusoidals**

$$\sum_{k=0}^{N} a_k \cos(\omega_k n + \phi_k) \leftrightarrow \sum_{k=0}^{N} \pi a_k e^{j\phi_k}\delta(\omega - \omega_k) + \pi a_k e^{-j\phi_k}\delta(\omega + \omega_k)$$

For simplicity, each transform is represented over only one period; for generality, phase offsets are included.

## 2.5 Uncertainty Principle

We saw in Example 2.3 a fundamental property of the Fourier transform pair: A signal cannot be arbitrarily narrow in time and in frequency. We saw in Figure 2.2 that the Fourier transform increased in spread as the time sequence decreased in width. This property can be stated more precisely in the *uncertainty principle*. To do so requires a formal definition of the width of the signal and its Fourier transform. We refer to these signal characteristics as *duration*, denoted by $D(x)$ and *bandwidth*,[4] denoted by $B(x)$, and define them respectively as

$$\begin{aligned} D(x) &= \sum_{n=-\infty}^{\infty} (n - \bar{n})^2 |x[n]|^2 \\ B(x) &= \int_{-\pi}^{\pi} (\omega - \bar{\omega})^2 |X(\omega)|^2 d\omega \end{aligned} \tag{2.4}$$

where $\bar{n}$ is the *average time* of the signal, i.e., $\bar{n} = \sum_{n=-\infty}^{\infty} n|x[n]|^2$, and $\bar{\omega}$ is its *average frequency*, i.e, $\bar{\omega} = \int_{-\pi}^{\pi} \omega |X(\omega)|^2 d\omega$ [2]. In these definitions, in order that the time and frequency averages be meaningful, we assume that the signal energy is unity, i.e., $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 1$ or that the signal has been normalized by its energy. These duration and bandwidth values give us a sense of the concentration of a signal, or of its Fourier transform, about its average location. The definitions of *signal* or *transform width* are motivated by the definition of the variance, or "spread," of a random variable. In fact,

[4] A more traditional definition of *bandwidth*, not necessarily giving the same value as that in Equation (2.4), is the distance between the 3 dB attenuation points around the average frequency.