how we define the complex logarithm.[2] For the trivial case of real and positive z-transforms, the logarithm, sometimes referred to as the "real logarithm," of the product is the product of the logarithms. Generally, however, this property is more difficult to obtain, as illustrated in the following example:

EXAMPLE 6.4 Consider two real and positive values a and b. Then $\log(ab) = \log(a) + \log(b)$. On the other hand, if $b < 0$, then $\log(ab) = \log(a|b|e^{jk\pi}) = \log(a) + \log(|b|) + jk\pi$, where k is an odd integer. Thus, the definition of $\log(ab)$ in this case is ambiguous. ▲

Example 6.4 indicates that special consideration must be made in defining the logarithm operator for complex $X(z)$ in order to make the logarithm of a product the sum of the logarithms [13]. Suppose for simplicity that we evaluate $X(z)$ on the unit circle ($z = e^{j\omega}$), i.e., we evaluate the Fourier transform.[3] Then we consider the real and imaginary parts of the complex logarithm by writing the logarithm in polar form as

$$\begin{aligned}\log[X(\omega)] &= \log(|X(\omega)|e^{j\angle X(\omega)}) \\ &= \log(|X(\omega)|) + j\angle X(\omega).\end{aligned} \tag{6.5}$$

Then, if $X(\omega) = X_1(\omega)X_2(\omega)$, we want the logarithm of the real parts and the logarithm of the imaginary parts to equal the sum of the respective logarithms. The real part is the logarithm of the magnitude and, for the product $X_1(\omega)X_2(\omega)$, is given by

$$\begin{aligned}\log(|X(\omega)|) &= \log(|X_1(\omega)X_2(\omega)|) \\ &= \log(|X_1(\omega)|) + \log(|X_2(\omega)|)\end{aligned} \tag{6.6}$$

provided that $|X_1(\omega)| > 0$ and $|X_2(\omega)| > 0$, which is satisfied when zeros and poles of $X(z)$ do not fall on the unit circle. In this case, there is no problem with the uniqueness and "additivity" of the logarithms. The imaginary part of the logarithm is the phase of the Fourier transform and requires more careful consideration. As with the real part, we want the imaginary parts to add

$$\begin{aligned}\angle X(\omega) &= \angle[X_1(\omega)X_2(\omega)] \\ &= \angle X_1(\omega) + \angle X_2(\omega).\end{aligned} \tag{6.7}$$

The relation in Equation (6.7), however, generally does not hold due to the ambiguity in the definition of phase, i.e., $\angle X(\omega) = \mathrm{PV}[\angle X(\omega)] + 2\pi k$, where k is any integer value, and where PV denotes the principle value of the phase which falls in the interval $[-\pi, \pi]$. Since an arbitrary multiple of 2π can be added to the principal phase values of $X_1(\omega)$ and $X_2(\omega)$, the additivity property generally does not hold. One approach to obtain uniqueness is to force continuity within the definition of phase, i.e., select the integer k such that the function $\angle X(\omega) = \mathrm{PV}[\angle X(\omega)] + 2\pi k$ is continuous (Figure 6.5). Continuity ensures not only uniqueness, but

[2] There is no such problem with the inverse exponential operator since $e^{a+b} = e^a e^b$. Thus, along with the forward and inverse z-transforms, addition is unambiguously mapped back to convolution.

[3] We assume the sequence $x[n]$ is stable and thus that the region of convergence of $X(z)$ includes the unit circle.