Our objective is to estimate the numerator and denominator coefficients b_k and a_k, respectively. One approach is to minimize the mean-squared error between the *model* and the *measurement*, i.e., minimize $\sum_{n=-\infty}^{\infty}(x[n] - s[n])^2$ with respect to the unknown coefficients. Using Parseval's relation, we can map this error function to the frequency domain in terms of $A(\omega)$ and $B(\omega)$. Minimizing this error with respect to the unknown polynominal coefficients leads to equations that are nonlinear in these parameters and thus difficult to solve (Exercise 5.21) [16].

5.7.1 Linearization

An alternative approach to solving for the above polynomial coefficients is to linearize the parameter estimation problem by first cross-multiplying the model in Equation (5.45) to obtain

$$X(z)A(z) = B(z) \tag{5.46}$$

which, in the time domain, is given by the difference equation

$$x[n] - \sum_{k=1}^{p} a_k x[n-k] = \sum_{k=0}^{q} b_k \delta[n-k]$$

from which we see that

$$x[n] - \sum_{k=1}^{p} a_k x[n-k] = 0, \quad n > q. \tag{5.47}$$

We can, therefore, set up an error minimization, involving the model $x[n]$ and the measurement $s[n]$, that solves for the predictor coefficients in Equation (5.47) and that is identical to the all-pole case. Specifically, let $\tilde{s}[n]$ be the prediction of the measurement $s[n]$, i.e., $\tilde{s}[n] = \sum_{k=1}^{p} \alpha_k s[n-k]$. We then minimize the error function $\sum_{n=q+1}^{\infty}(s[n] - \tilde{s}[n])^2$ with respect to the coefficients α_k. We have assumed here the correct model order p; in practice, we would set p equal to twice the expected number of resonances for a particular bandwidth. Let us now think of the estimated coefficients of $A(z)$ as a time sequence $-\alpha_n$ for $n = 0, 1, 2 \ldots p$ ($\alpha_o = -1$). With Equation (5.46) as the basis, we then "inverse filter" the measurement $s[n]$ by $-\alpha_n$ to obtain an estimate of the numerator "sequence," i.e.,

$$\beta_n = s[n] * (-\alpha_n) \tag{5.48}$$

where we denote the estimates of b_n, the coefficients of the numerator polynomial $B(z)$, by β_n. We have therefore separated the pole estimation problem from the zero estimation problem while keeping the solution linear in the unknown parameters.

In practice, however, the inverse filtering of $s[n]$ does not lead to reliable estimates of the numerator coefficients, and thus the true zeros of the underlying signals. This is because the method relies on an accurate estimate of $A(z)$ and because it assumes the relation $S(z)A(z) = B(z)$, both of which, in reality, suffer from inexactness. More reliable estimates rely on formulating a least-squared error estimation for the unknown b_n's. In one such approach, the impulse response corresponding to the estimated all-pole component of the system function, i.e., $\frac{1}{\hat{A}(z)} = \frac{1}{1-\sum_{k=1}^{p}\alpha_k z^{-1}}$, is first obtained. Denoting this impulse response by $h_{\underline{\alpha}}[n]$, we then