which can be shown to be periodic with period $\frac{2\pi}{2\tau}$ (Figure 4.17a):

$$\begin{aligned} V_a(\Omega + \frac{2\pi}{2\tau}) &= \sum_{k=0}^{\infty} b_k e^{-j(\Omega + \frac{2\pi}{2\tau})k2\tau} \\ &= \sum_{k=0}^{\infty} b_k e^{-j\Omega k2\tau} e^{-j(\frac{2\pi}{2\tau})k2\tau} \\ &= V_a(\Omega). \end{aligned}$$

The intuition for this periodicity is that we have "discretized" the continuous-space tube with space-interval $\Delta x = \frac{l}{N}$, and the corresponding time-interval $\tau = \frac{\Delta x}{c}$, so we expect periodicity to appear in the transfer function representation.

From Figure 4.17a, we see that $V_a(\Omega)$ has the form of a Fourier transform of a sampled continuous waveform with sampling time interval $T = 2\tau$. We use this observation to transform the analog filtering operation to discrete-time form with the following steps illustrated in Figure 4.17:

S1: Using the impulse-invariance method, i.e., replacing e^{sT} with the complex variable z where $T = 2\tau$, we transform the system function $V_a(\Omega)$ to discrete-time:

$$V_a(s) = \sum_{k=0}^{\infty} b_k (e^{s2\tau})^{-k}$$

which, with the replacement of e^{sT} by the complex variable z, becomes

$$V(z) = \sum_{k=0}^{\infty} b_k z^{-k}.$$

The frequency response $V(\omega) = V(z)|_{z=e^{j\omega}}$ is designed to match desired formant resonances over the interval $[-\pi, \pi]$.

S2: Consider an excitation function $u_g(t)$ that is bandlimited with maximum frequency $\Omega_{max} = \frac{\pi}{2\tau}$ and sampled with a periodic impulse train with sampling interval $T = 2\tau$, thus meeting the Nyquist criterion to avoid aliasing. The Fourier transform of the resulting excitation is denoted by $U_g(\Omega)$. We then convert the impulse-sampled continuous-time input to discrete time. This operation is illustrated in the frequency-domain in Figure 4.17b.

S3: A consequence of using the impulse invariance method to perform filter conversion is a straightforward conversion of a continuous-time flow graph representation of the model to a discrete-time version. An example of this transformation is shown in Figure 4.18 for the two-tube case. Since the mapping of e^{sT} to z yields $V(z)$, the discrete-time signal flow graph can be obtained in a similar way. A delay of τ seconds corresponds to the continuous-time factor $e^{-s\tau} = e^{-s2\tau/2} = e^{-sT/2}$, which in discrete-time is a half-sample delay. Therefore, in a signal flow graph we can replace the delay τ by $z^{-1/2}$. Since a half-sample delay is difficult to implement (requiring interpolation), we move all lower-branch delays to