where, because the corresponding response is real, the poles occur in complex conjugate pairs. There are then an infinite number of poles that occur on the $j\Omega$ axis and can be written as

$$s_n = \pm j\left[\frac{(2k+1)\pi c}{2l}\right], \qquad k = 0, \pm 1, \pm 2, \ldots$$

corresponding to the *resonant frequencies* of the uniform tube. The poles fall on the $j\Omega$ axis, and thus the frequency response at resonant frequencies is infinite in amplitude, because there is no energy loss in the simple configuration. In particular, it was assumed that friction of air particles does not occur along the vocal tract walls, that heat is not conducted away through the vocal tract walls, and that the walls are rigid and so energy was not lost due to their movement. In addition, because we assumed boundaries consisting of an ideal volume velocity source and zero pressure at the lips, no energy loss was entailed at the input or output to the uniform tube. In practice, on the other hand, such losses do occur and will not allow a frequency response of infinite amplitude at resonance. Consider by analogy a lumped electrical circuit with a resistor, capacitor, and inductor in series. Without the resistor to introduce thermal loss into the network, the amplitude of the frequency response at resonance equals infinity. We investigate these forms of energy loss in the following two sections. Energy loss may also occur due to rotational flow, e.g., vortices and other nonlinear effects that were discarded in the derivation of the wave equation, such as from Equation (4.3). These more complex forms of energy loss are discussed in Chapter 11.

4.3.2 Effect of Energy Loss

Energy loss can be described by differential equations *coupled* to the wave equation that describes the pressure/volume velocity relations in the lossless uniform tube. These coupled equations are typically quite complicated and a closed-form solution is difficult to obtain. The solution therefore is often found by a numerical simulation, requiring a discretization in time and in space along the x variable.

Wall Vibration — Vocal tract walls are pliant and so can move under pressure induced by sound propagation in the vocal tract. To predict the effect of wall vibration on sound propagation, we generalize the partial differential equations of the previous section to a tube whose cross-section is nonuniform and time-varying. Under assumptions similar to those used in the derivation of Equation (4.7), as well as the additional assumption that the cross-section of a nonuniform tube does not change "too rapidly" in space (i.e., the x direction) and in time, Portnoff [26] has shown that sound propagation in a nonuniform tube with time- and space-varying cross-section $A(x,t)$ is given by

$$\begin{aligned} -\frac{\partial p}{\partial x} &= \rho \frac{\partial (u/A)}{\partial t} \\ -\frac{\partial u}{\partial x} &= \frac{1}{\rho c^2}\frac{\partial pA}{\partial t} + \frac{\partial A}{\partial t} \end{aligned} \tag{4.20}$$

which, for a uniform time-invariant cross-section, reduces to the previous Equation pair (4.1), (4.4).