CHAPTER 31

Robust Design

Customers expect excellent performance from the products they purchase, and producers of products expect excellent process performance in the production of those products. Performance excellence includes at least two dimensions: a high average level of performance and consistent performance at that level. In statistical terms, this means an average performance that is close to target with low variation about that average.

Some causes of variation are identifiable, and it is often possible to quantify the contributions of those causes to the overall variation in product or process performance. Overall variation may also change with the settings of controllable inputs that impact the performance. In such cases, it may be possible to identify settings of controllable inputs that minimize overall variation in performance while ensuring that the level of performance is close to its target. Achievement of those dual objectives is the role of *Robust Design*.[1, 2] Data for this chapter are contained in the Minitab Project file Chapter 31—Robust Design. MPJ.

Quantifying Robust Design Performance

Robust design is particularly important in ensuring that the performance of a product will be consistently good over a wide range of conditions of use. For example, an automobile should perform well at extremely high and extremely low temperatures, as well at normal temperatures. It should also perform equally as well in extremely dry and extremely wet climates, as well as in normal climates. The dual objectives of:

a) achieving average performance close to target, and
b) achieving low variation about that target

1. G. E. P. Box, J. S. Hunter, and W. G. Hunter, *Statistics for Experimenters, 2nd ed*. (New York: John Wiley, 2005).
2. R. H. Myers and D. C. Montgomery, *Response Surface Methodology, 2nd ed.* (New York: John Wiley, 2002).

can be evaluated by the Mean Squared Deviation (MSD)

$$MSD = \frac{\sum_{i=1}^{n} (y_i - \text{target})^2}{n}$$

where y_i is the measured performance value for observation i
and n is the number of observations.

MSD can be divided into two components, as demonstrated by the equation:

$$MSD = (\bar{y} - \text{target})^2 + \frac{n-1}{n} s^2$$

where $(\bar{y} - \text{target})^2$ is a measure of the departure of performance from target, and

$\frac{n-1}{n} s^2$ is a measure of the variation about the average performance.

Minimizing these two components of MSD requires the following:

Adjusting the settings of controllable inputs to center the performance of a product or process at its target value

Adjusting the settings of controllable inputs to minimize the variation in performance of a product or process about its average value.

Using Critical Input Variables to Improve Robustness

Selection of the appropriate adjustments to achieve both of these objectives requires that we accomplish the following two tasks:

Identification of the controllable inputs that influence the *average performance* and an equation describing the relationship between average performance and those controllable inputs.

Identification of the controllable inputs that influence the *variation in performance* and an equation describing the relationship between variation in performance and those controllable inputs.

A robust design study can identify the critical controllable input variables for each of these tasks and provide the data required to develop an equation that describes the required relationship for each task. Once equations have been developed for average performance and variation in performance, one strategy for achieving the dual objectives uses the following steps:

Bring average performance as close to target as possible using controllable inputs that impact average performance but not variation in performance. If no such controllable inputs can be identified, use those controllable inputs that affect both average performance and variation in performance but that have the smallest effects on variation in performance.

Reduce the variation in performance as much as possible using controllable inputs that affect variation in performance but not average performance. If no such controllable inputs can be identified, use those controllable inputs that were not used to minimize departure of average performance from target in step 1.

Set controllable inputs that impact neither average performance nor variation in performance at their most cost-effective settings.

Carry out a capability study of performance at the recommended settings of all controllable inputs.

If the controllable inputs that influence average performance are different from the controllable inputs that influence variation in performance, the two objectives may be achieved independently of one another. In many cases, however, some controllable input variables affect both average performance and variation in performance, thereby creating a potential conflict between the two objectives. In such cases, a compromise optimum may be required using multiple response optimization, which is described in Chapter 33.

The Taguchi Approach to Robust Design

Genichi Taguchi deserves credit for emphasizing the importance of considering variation in performance as well as average performance. His approach to robust design involves experiments that include both controllable inputs and noise variables. The noise variables, which are expected to affect variation in performance, are actually controlled in Taguchi's experiments, which may be possible in a laboratory setting.

A Taguchi robust design consists of an inner array involving only controllable inputs and an outer array involving only noise variables. All of the runs in the outer array are conducted at each combination of settings of the controllable inputs in the inner array, thereby creating an overall crossed array design that potentially can involve a large number of runs. A simple example will illustrate this arrangement.

The Inner and Outer Arrays

Consider a situation in which five controllable inputs, X1, X2, X3, X4, and X5, and two noise variables, Z1 and Z2, are to be investigated. The inner array, involving only X1, X2, X3, X4, and X5, could be what Taguchi calls an L8 array, shown in Table 31–1 (Taguchi uses 1 and 2, instead of -1 and 1, to denote the low and high levels of each factor). This array is a 2^{5-2} fractional factorial design in which each main effect is aliased with one or more 2-factor interaction effects.

The outer array, involving only Z1 and Z2, could be a 2-level full factorial design, as shown in Table 31–2.

Table 31–1 Inner Array

X_1	X_2	X_3	X_4	X_5
–1	–1	–1	–1	–1
–1	–1	–1	1	1
–1	1	1	–1	–1
–1	1	1	1	1
1	–1	1	–1	1
1	–1	1	1	–1
1	1	–1	–1	1
1	1	–1	1	–1

Table 31–2 Outer Array

Z_1	Z_2
–1	–1
–1	1
1	–1
1	1

The complete design is shown in Table 31–3.

The mean and the standard deviation of performance at each combination of settings for the five controllable inputs can be calculated from the four runs at that combination of settings. It is then possible to identify controllable inputs that impact:

Average performance but not variation in performance
Variation in performance but not average performance
Both average performance and variation in performance
Neither average performance nor variation in performance.

Equations can be developed that relate:

Average performance to controllable inputs
Variation in performance to controllable inputs.

To draw valid inferences about the significance of individual terms in the latter equation, the logarithm (base e or base 10) of the standard deviation should be used as the response variable instead of the standard deviation itself.

Table 31–3 Combined Inner and Outer Arrays

X_1	X_2	X_3	X_4	X_5	Z_1	Z_2
–1	–1	–1	–1	–1	–1	–1
–1	–1	–1	–1	–1	–1	1
–1	–1	–1	–1	–1	1	–1
–1	–1	–1	–1	–1	1	1
–1	–1	–1	1	1	–1	–1
–1	–1	–1	1	1	–1	1
–1	–1	–1	1	1	1	–1
–1	–1	–1	1	1	1	1
–1	1	1	–1	–1	–1	–1
–1	1	1	–1	–1	–1	1
–1	1	1	–1	–1	1	–1
–1	1	1	–1	–1	1	1
–1	1	1	1	1	–1	–1
–1	1	1	1	1	–1	1
–1	1	1	1	1	1	–1
–1	1	1	1	1	1	1
1	–1	1	–1	1	–1	–1
1	–1	1	–1	1	–1	1
1	–1	1	–1	1	1	–1
1	–1	1	–1	1	1	1
1	–1	1	1	–1	–1	–1
1	–1	1	1	–1	–1	1
1	–1	1	1	–1	1	–1
1	–1	1	1	–1	1	1
1	1	–1	–1	1	–1	–1
1	1	–1	–1	1	–1	1
1	1	–1	–1	1	1	–1
1	1	–1	–1	1	1	1
1	1	–1	1	–1	–1	–1
1	1	–1	1	–1	–1	1
1	1	–1	1	–1	1	–1
1	1	–1	1	–1	1	1

Some Limitations

There are some limitations in using this approach to robust design. One obvious challenge is the large number of runs required for even modest numbers of controllable inputs and noise variables. Another is the aliasing among individual and joint effects of the controllable inputs. And finally, if the experiment is carried out by conducting all of the runs at each combination of settings of the controllable inputs before changing to another combination of settings of the controllable inputs, this design becomes a split plot design and the analysis of results must be carried out in an appropriate manner. An alternative and more efficient approach to robust design, using response surface designs that include both controllable inputs and noise variables as factors, is described later in this chapter.

Robust Design Example

The five-factor example used in Chapter 30 to illustrate a fractional factorial design can also be used here to demonstrate the Taguchi procedure for robust design. The five controllable inputs are shown in Table 31–4. The performance variable is film thickness, for which the target value is 1.0 mil.

Let's consider in addition two noise variables, relative humidity (low or high) denoted by Z1, and operating shift (1 or 2) denoted by Z2. The results from the 32-run design just presented in Table 31–3 are shown in Table 31–5.

Analyzing Average Performance

Taguchi advocates main effects plots for identifying controllable inputs that influence average performance. Main effects plots for average performance for this example are shown in Figure 31–1.

These plots suggest that X2 (Supplier) has the largest effect on average performance (average film thickness), and that average performance is higher with Supplier 2 than with Supplier 1. Unfortunately, the target film thickness is almost exactly halfway between the average thickness obtained with material from Supplier 1 and the average thickness obtained with material from Supplier 2. In Minitab, clicking **Stat > DOE Factorial > Analyze Factorial Design** allows

Table 31–4 Controllable Inputs

Factor	Low Setting	High Setting
$\mathbf{X_1}$ % Additive	10	15
$\mathbf{X_2}$ Supplier	1	2
$\mathbf{X_3}$ Production Rate (rolls/hr)	100	120
$\mathbf{X_4}$ Temperature (°F)	140	180
$\mathbf{X_5}$ Belt Speed (ft/min)	25	50

Table 31–5 Experimental Results from Combined Inner and Outer Arrays

					Z_1	−1	−1	1	1			
X_1	X_2	X_3	X_4	X_5	Z_2	−1	1	−1	1	$\bar{y}$	s	ln(s)
−1	−1	−1	−1	−1		0.92	0.86	0.83	0.84	0.862	0.0403	−3.21
−1	−1	−1	1	1		0.76	0.74	0.72	0.71	0.732	0.0222	−3.81
−1	1	1	−1	−1		0.96	0.89	0.95	0.94	0.935	0.0311	−3.47
−1	1	1	1	1		1.27	1.28	1.25	1.27	1.268	0.0126	−4.38
1	−1	1	−1	1		0.81	0.83	0.86	0.89	0.847	0.035	−3.35
1	−1	1	1	−1		0.92	0.88	0.95	0.94	0.923	0.031	−3.48
1	1	−1	−1	1		0.98	1.03	1.05	1	1.015	0.0311	−3.47
1	1	−1	1	−1		1.44	1.4	1.48	1.45	1.443	0.033	−3.41

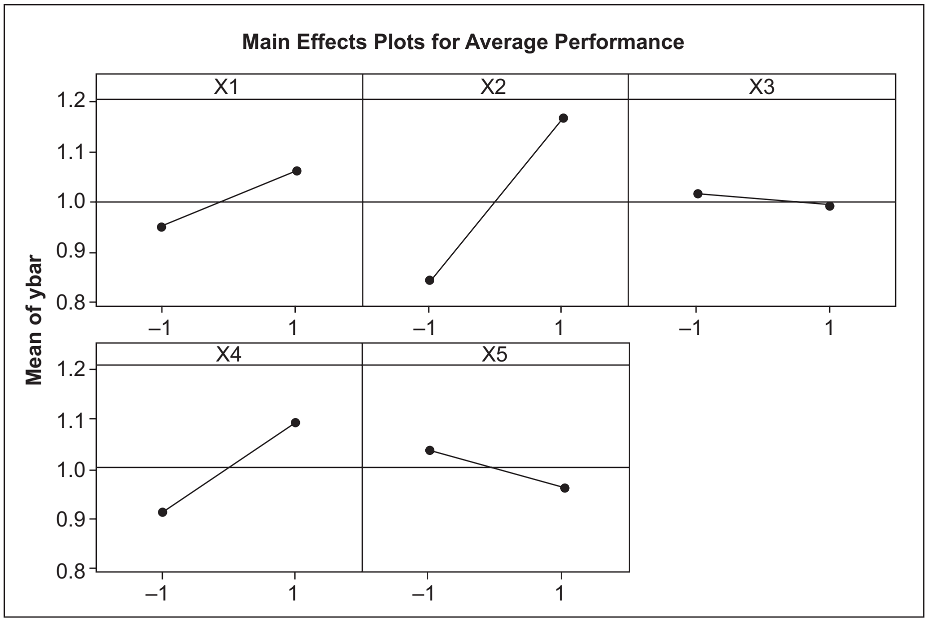

Figure 31–1 Main Effects Plots for Performance

assessment of the statistical significance of this effect as well as other main effects and interaction effects, with the results shown in Figure 31–2 (after deleting nonsignificant combinations of interaction effects).

The main effect of the supplier is statistically significant at a significance level of 0.10, but so is the (Supplier)(Temperature) (i.e., X2X4) interaction effect, which is aliased with the (Production

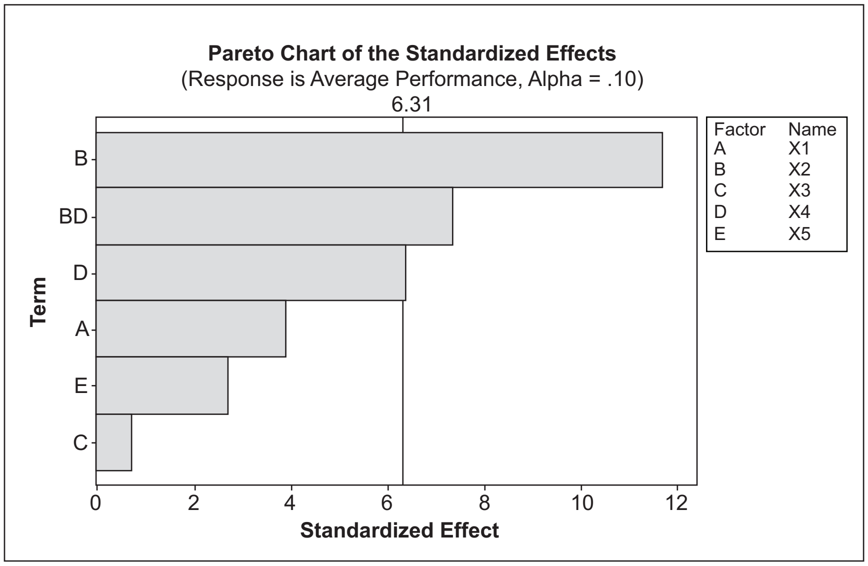

Figure 31–2 Pareto Summary of Effects for Average Performance

Rate)(Belt Speed) (X3X5) interaction effect. The main effect of Temperature (X4) is marginally significant at this significance level. This analysis demonstrates the danger of relying on only main effects to identify influential inputs. All other effects, each of which is aliased with three other effects, are indicated to have no significant influences on average performance.

An interaction plot involving the supplier and temperature (X2 and X4), shown in Figure 31–3, reveals that the preferred conditions for those two controllable inputs for achieving the target thickness are with material from Supplier 2 at temperature 140 °F, conditions that are not apparent from the main effects plots alone.

Analyzing Variation in Performance

A similar analysis performed on ln(s) produces the results shown in Figure 31–4.

These main effects plots suggest that all (or none) of the controllable inputs may influence variation in performance (variation in film thickness). The statistical significance (at the 0.10 level) of the main effect of each of the five controllable inputs is shown in Figure 31–5. The interaction effects that are not aliased with main effects are found to be nonsignificant.

The summary indicates that only three of the controllable inputs, Temperature (X4), Belt Speed (X5), and % Additive (X1), significantly influence the variation in performance.

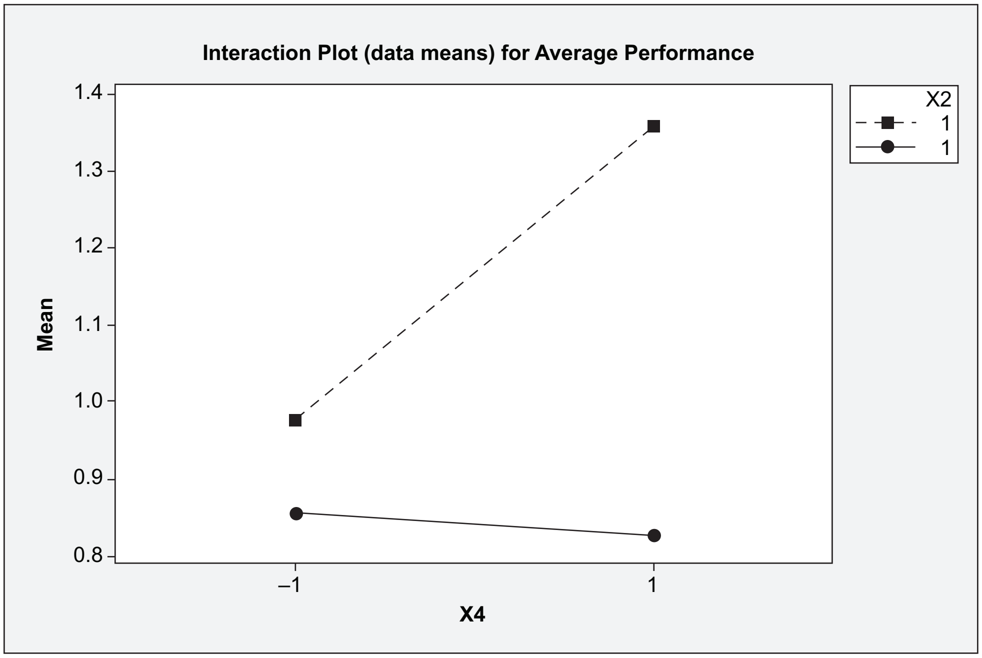

Figure 31–3 (Supplier)(Temperature) (X2)(X4) Interaction Plot

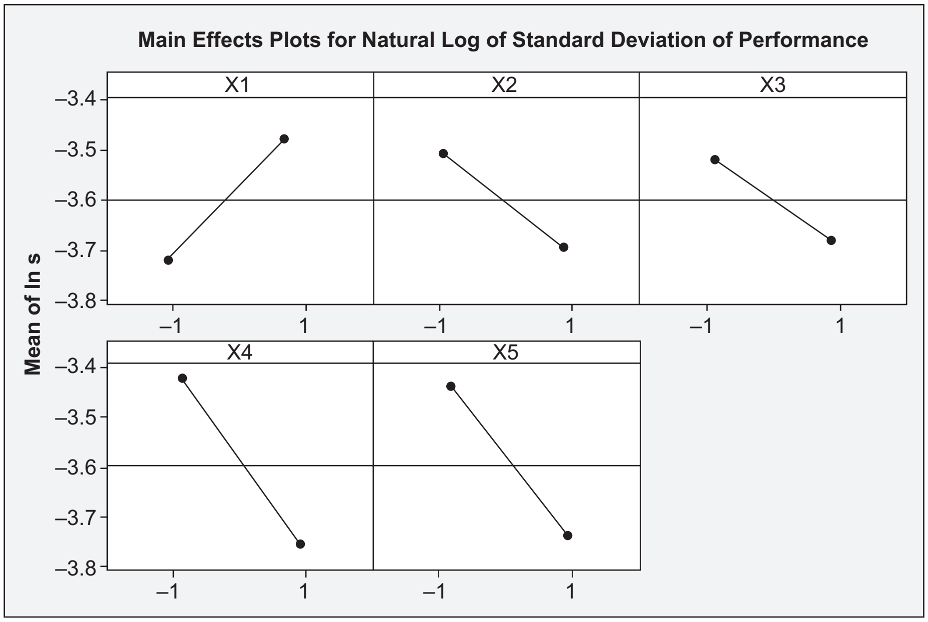

Figure 31–4 Main Effects Plots for Loge Standard Deviation of Performance

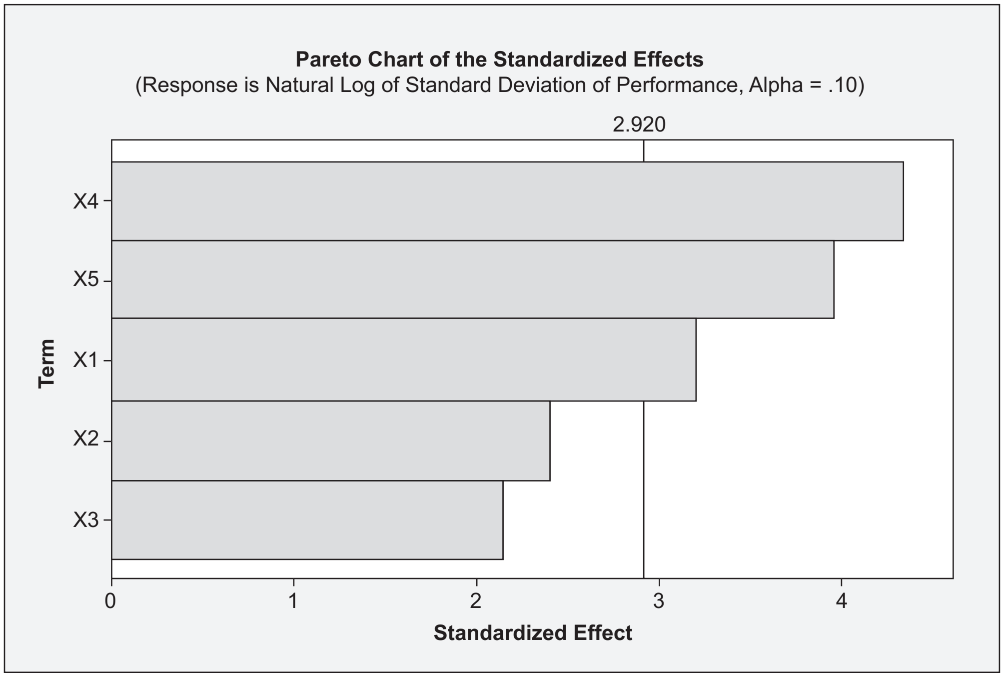

Figure 31–5 Pareto Summary of Effects for Loge Standard Deviation of Performance

From these results, it appears that average film thickness = 1.0 mil could be achieved with low variation in performance with the following settings:

$\mathbf{X_1}$ % Additive	10	
$\mathbf{X_2}$ Supplier	2	
$\mathbf{X_3}$ Production Rate (rolls/hr)	120	(as high as possible because it has no apparent effect on average performance or variation in performance)
$\mathbf{X_4}$ Temperature (°F)	140	
$\mathbf{X_5}$ Belt Speed (ft/min)	50	

However, there is clearly a conflict concerning the best setting for temperature (x_4). To achieve an average film thickness close to target, the best temperature is 140°F, but to achieve minimum variation in film thickness, the temperature should be 180°F. An optimal compromise solution can be found using Multiple Response optimization, which is discussed in Chapter 33.+

Alternative Approaches to Robust Design

Any design that includes replicate runs at each combination of settings of the controllable inputs can be used as a robust design, even if the noise variables creating the variation among the replicate runs are unknown. The analysis of results is similar to that just described for the data in Table 31–5. The mean response and the variation in response, evaluated using the logarithm of the standard deviation of the response values, are calculated for each combination of settings of the controllable inputs. The type of design used for the controllable inputs will determine the degree of detail provided to describe the individual and joint influences of the controllable inputs on the average response and variation in response.

Borror, Montgomery, and Myers (2002)[3] proposed the use of response surface designs as robust designs in which both the controllable inputs and the noise variables are treated as factors. This approach requires the noise variables to be controlled at specified settings, which may be possible in a special environment such as a laboratory. The response surface approach is now illustrated using a modified version of the candy wrapper example employed in the preceding section of this chapter.

Robust Design with Response Surface Techniques

This illustration involves three controllable inputs over the ranges shown in Table 31–6 and two noise variables over the ranges shown in Table 31–7.

The performance variable is, again, film thickness. The objective is to achieve a film thickness of 1.00 with minimum variation.

As described in Chapter 30, Minitab offers two types of response surface designs, Central Composite Designs and Box-Behnken Designs. For five factors (three controllable inputs and two noise variables), the 32-run Central Composite Design shown in Table 31–8 is the more economical choice.

Additional economy can be realized by omitting the axial points (shown in italics in Table 31–8) for the two noise variables. The effect of these deletions is to give up information

Table 31–6 Controllable Inputs for Response Surface Design

Controllable Input	Range of Interest		
% Additive	5%	to	25%
Temperature	160 °F	to	200 °F
Belt Speed	50 ft/min	to	70 ft/min

Table 31–7 Noise Variables for Response Surface Design

Noise Variable	Range of Interest		
Relative Humidity	50%	to	70%
Ambient Particulate Level	1%	to	5%

3. C. M. Borror, D. C. Montgomery, and R. H. Myers, "Evaluation of Statistical Designs for Experiments Involving Noise Variables," *Journal of Quality Technology, Vol. 34, No. 1*, (2002), p. 54–70.

Table 31–8 32-Run Central Composite Design

% Additive	Temperature	Belt Speed	Relative Humidity	Particulate Level
10	170	55	55.0	4
20	170	55	55.0	2
10	190	55	55.0	2
20	190	55	55.0	4
10	170	65	55.0	2
20	170	65	55.0	4
10	190	65	55.0	4
20	190	65	55.0	2
10	170	55	70.0	2
20	170	55	70.0	4
10	190	55	70.0	4
20	190	55	70.0	2
10	170	65	70.0	4
20	170	65	70.0	2
10	190	65	70.0	2
20	190	65	70.0	4
5	180	60	62.5	3
25	180	60	62.5	3
15	160	60	62.5	3
15	200	60	62.5	3
15	180	50	62.5	3
15	180	70	62.5	3
15	*180*	*60*	*47.5*	*3*
15	*180*	*60*	*77.5*	*3*
15	*180*	*60*	*62.5*	*1*
15	*180*	*60*	*62.5*	*5*
15	180	60	62.5	3
15	180	60	62.5	3
15	180	60	62.5	3
15	180	60	62.5	3
15	180	60	62.5	3
15	180	60	62.5	3

about the quadratic effects of the noise variables, which is not a serious loss. The resulting design is shown in Table 31–9 with the measured film thickness for each run. The order of execution of the runs is randomized.

The settings of the five factors are coded as follows:

$$A = (\% \text{ Additive} - 15)/5$$

$$B = (\text{Temperature} - 180)/10$$

$$C = (\text{Belt Speed} - 60)/5$$

$$Z_1 = (\text{Relative Humidity} - 62.5)/7.5$$

$$Z_2 = (\text{Ambient Particulate Level} - 3)/1$$

Table 31–9 Response Surface Design with Experimental Results

%Additive	Temp	Belt Speed	Rel.Humidity	Particulate	Thickness
10	170	55	55.0	4	1.03
20	170	55	55.0	2	0.87
10	190	55	55.0	2	1.26
20	190	55	55.0	4	1.08
10	170	65	55.0	2	0.83
20	170	65	55.0	4	1.47
10	190	65	55.0	4	0.94
20	190	65	55.0	2	1.03
10	170	55	70.0	2	0.75
20	170	55	70.0	4	0.78
10	190	55	70.0	4	1.24
20	190	55	70.0	2	1.31
10	170	65	70.0	4	1.11
20	170	65	70.0	2	0.44
10	190	65	70.0	2	1.06
20	190	65	70.0	4	0.93
5	180	60	62.5	3	1.15
25	180	60	62.5	3	1.17
15	160	60	62.5	3	0.45
15	200	60	62.5	3	0.95
15	180	50	62.5	3	0.99
15	180	70	62.5	3	0.91
15	180	60	62.5	3	1.15
15	180	60	62.5	3	1.16
15	180	60	62.5	3	1.14
15	180	60	62.5	3	1.21
15	180	60	62.5	3	1.18
15	180	60	62.5	3	1.11

and a full quadratic equation (without quadratic terms in the two noise variables) is fitted to the coded data. After deleting nonsignificant terms, the fitted equation for measured film thickness, expressed in the coded units of the five factors, is:

$$\begin{aligned}\text{Predicted Thickness} = {} & 1.165 - 0.011\,A + 0.107\,B - 0.028\,C - 0.113\,B^2 - 0.051\,C^2 \\ & - 0.084\,BC - 0.056\,Z_1 + 0.064\,Z_2 - 0.068\,AZ_1 + 0.084\,B\,Z_1 \\ & - 0.123\,B\,Z_2 - 0.036\,C\,Z_1 + 0.072\,C\,Z_2\end{aligned}$$

Minitab's output for this fitted model is shown in Figure 31–6.

The corresponding fitted equation expressed in the original units of the five factors is:

$$\begin{aligned}\text{Predicted Thickness} = {} & 1.165 - 0.005\,(\%\ \text{Add}) + 0.054\,(\text{Temp}) - 0.014\,(\text{Speed}) \\ & - 0.028\,(\text{Temp})^2 - 0.013\,(\text{Speed})^2 - 0.021\,(\text{Temp})(\text{Speed}) \\ & - 0.056\,(\text{Humidity}) + 0.064\,(\text{Particulate})\end{aligned}$$

$$- 0.034\ (\%\ \text{Add})(\text{Humidity}) + 0.042\ (\text{Temp})(\text{Humidity})$$
$$- 0.062\ (\text{Temp})(\text{Particulate}) - 0.018\ (\text{Speed})(\text{Humidity})$$
$$+ 0.036\ (\text{Speed})(\text{Particulate})$$

Response Surface Regression: thickness versus A, B, C, Z1, Z2

The analysis was done using coded units.

Estimated Regression Coefficients for thickness

Term	Coef	SE Coef	T	P
Constant	1.16525	0.011794	98.796	0.000
A	-0.01125	0.007613	-1.478	0.162
B	0.10708	0.007613	14.065	0.000
C	-0.02792	0.007613	-3.667	0.003
Z1	-0.05562	0.009324	-5.966	0.000
Z2	0.06437	0.009324	6.904	0.000
B*B	-0.11306	0.007223	-15.654	0.000
C*C	-0.05056	0.007223	-7.001	0.000
A*Z1	-0.06813	0.009324	-7.306	0.000
B*C	-0.08438	0.009324	9.049	0.000
B*Z1	0.08437	0.009324	9.049	0.000
B*Z2	-0.12313	0.009324	-13.205	0.000
C*Z1	-0.03562	0.009324	-3.821	0.002
C*Z2	0.07187	0.009324	7.708	0.000

S = 0.03730 R-Sq = 98.7% R-Sq(adj) = 97.4%

Figure 31–6 Analysis of Data in Table 31–9

To use the fitted equation in the coded factors, A, B, C, Z1 and Z2 as a predictor, the controllable inputs are regarded as fixed factors, whereas the noise variables are regarded as random variables, each with mean 0 and variances σ^2_{Z1} and σ^2_{Z2}, respectively. Estimates of the variances σ^2_{Z1} and σ^2_{Z2} must be available from previous experience. In fact, in this example the coded values of -1 and 1 for each coded noise variable are assumed to be about two standard deviations away from the average value of that noise variable.

Under these assumptions, the predictive equation for the average thickness consists of terms from the fitted equation in the coded factors that involve only the coded controllable inputs A, B, and C. In coded units this is:

$$\text{Predicted Thickness} = 1.165 - 0.011\ \text{A} + 0.107\ \text{B} - 0.028\ \text{C} - 0.113\ \text{B}^2 - 0.051\ \text{C}^2 - 0.084\ \text{BC}$$

An equation describing how the variance of the film thickness depends on the settings of the controllable inputs can also be obtained from the fitted equation in the coded factors by evaluating the variance of each term and adding an estimate of the random error variance from the fitted equation, in this case $s^2 = (0.0373)^2 = 0.00139$.

$$\begin{aligned}&\text{Estimated Variance (predicted thickness)}\\ &\quad = (-\,0.056 - 0.068\text{ A} + 0.084\text{ B} - 0.036\text{ C})^2\,\sigma^2_{Z1}\\ &\qquad + (0.064 - 0.123\text{ B} + 0.072\text{ C})^2\,\sigma^2_{Z2} + 0.00139\end{aligned}$$

If the coded values of -1 and 1 for each noise variable are assumed to span a range of approximately four standard deviations for that coded noise variable, then in this example $\sigma_{Z1} \approx 0.5$ and $\sigma_{Z2} \approx 0.5$. Using these estimates,

$$\begin{aligned}&\text{Estimated Variance of Predicted Thickness}\\ &\quad = (-\,0.056 - 0.068\text{ A} + 0.084\text{ B} - 0.036\text{ C})^2\,(0.5)^2\\ &\qquad + (0.064 - 0.123\text{ B} + 0.072\text{ C})^2\,(0.5)^2 + 0.00139\end{aligned}$$

The optimal solution is found by minimizing the estimated variance of the predicted thickness subject to the constraint that the predicted mean thickness = 1.00, the target value. A grid search over values of A, B, and C indicates an approximately optimal solution at the following settings:

A = -1.0 (% Additive = 10 %)

B = -0.6 (Temperature = 174 °F)

C = -1.0 (Belt Speed = 55 ft/min)

Dealing with Variation

Variation is a fact of life. Nevertheless, it is often possible to achieve consistently good performance in a product or process despite the effects of noise. A robust design can evaluate the contributions of individual noise variables and provide information to identify settings of controllable inputs that minimize variation in process and/or product performance produced by noise variables while maintaining average performance at a specified target value.