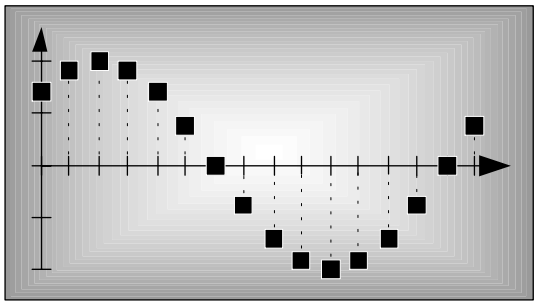
CHAPTER 1

Discrete Sequences and Systems



Digital signal processing has never been more prevalent or easier to perform. It wasn't that long ago when the fast Fourier transform (FFT), a topic we'll discuss in Chapter 4, was a mysterious mathematical process used only in industrial research centers and universities. Now, amazingly, the FFT is readily available to us all. It's even a built-in function provided by inexpensive spreadsheet software for home computers. The availability of more sophisticated commercial signal processing software now allows us to analyze and develop complicated signal processing applications rapidly and reliably. We can perform spectral analysis, design digital filters, develop voice recognition, data communication, and image compression processes using software that's interactive both in the way algorithms are defined and how the resulting data are graphically displayed. Since the mid-1980s the same integrated circuit technology that led to affordable home computers has produced powerful and inexpensive hardware development systems on which to implement our digital signal processing designs.[†] Regardless, though, of the ease with which these new digital signal processing development systems and software can be applied, we still need a solid foundation in understanding the basics of digital signal processing. The purpose of this book is to build that foundation.

In this chapter we'll set the stage for the topics we'll study throughout the remainder of this book by defining the terminology used in digital signal process-

[†] During a television interview in the early 1990s, a leading computer scientist stated that had automobile technology made the same strides as the computer industry, we'd all have a car that would go a half million miles per hour and get a half million miles per gallon. The cost of that car would be so low that it would be cheaper to throw it away than pay for one day's parking in San Francisco.

ing, illustrating the various ways of graphically representing discrete signals, establishing the notation used to describe sequences of data values, presenting the symbols used to depict signal processing operations, and briefly introducing the concept of a linear discrete system.

1.1 DISCRETE SEQUENCES AND THEIR NOTATION

In general, the term *signal processing* refers to the science of analyzing time-varying physical processes. As such, signal processing is divided into two categories, analog signal processing and digital signal processing. The term *analog* is used to describe a waveform that's continuous in time and can take on a continuous range of amplitude values. An example of an analog signal is some voltage that can be applied to an oscilloscope, resulting in a continuous display as a function of time. Analog signals can also be applied to a conventional spectrum analyzer to determine their frequency content. The term *analog* appears to have stemmed from the analog computers used prior to 1980. These computers solved linear differential equations by means of connecting physical (electronic) differentiators and integrators using old-style telephone operator patch cords. That way, a continuous voltage or current in the actual circuit was *analogous* to some variable in a differential equation, such as speed, temperature, air pressure, etc. (Although the flexibility and speed of modern-day digital computers have since made analog computers obsolete, a good description of the short-lived utility of analog computers can be found in reference [1].) Because present-day signal processing of continuous radio-type signals using resistors, capacitors, operational amplifiers, etc., has nothing to do with analogies, the term *analog* is actually a misnomer. The more correct term is *continuous signal processing* for what is today so commonly called analog signal processing. As such, in this book we'll minimize the use of the term *analog signals* and substitute the phrase *continuous signals* whenever appropriate.

The term *discrete-time signal* is used to describe a signal whose independent time variable is quantized so that we know only the value of the signal at discrete instants in time. Thus a discrete-time signal is not represented by a continuous waveform but, instead, a sequence of values. In addition to quantizing time, a discrete-time signal quantizes the signal amplitude. We can illustrate this concept with an example. Think of a continuous sinewave with a peak amplitude of 1 at a frequency f_o described by the equation

$$x(t) = \sin(2\pi f_o t) . \tag{1–1}$$

The frequency f_o is measured in hertz (Hz). (In physical systems, we usually measure frequency in units of hertz. One Hz is a single oscillation, or cycle, per second. One kilohertz (kHz) is a thousand Hz, and a megahertz (MHz) is

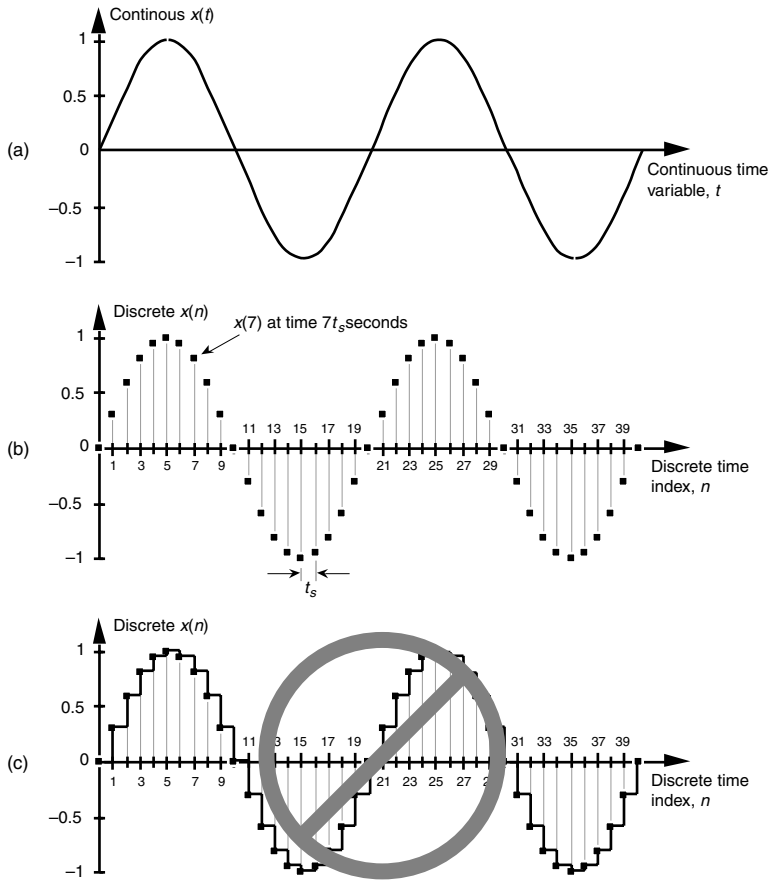


Figure 1–1 A time-domain sinewave: (a) continuous waveform representation; (b) discrete sample representation; (c) discrete samples with connecting lines.

one million Hz.[†]) With t in Eq. 1–1 representing time in seconds, the $f_o t$ factor has dimensions of cycles, and the complete $2\pi f_o t$ term is an angle measured in radians.

Plotting Eq. (1–1), we get the venerable continuous sinewave curve shown in Figure 1–1(a). If our continuous sinewave represents a physical volt-

[†] The dimension for frequency used to be *cycles/second*; that's why the tuning dials of old radios indicate frequency as kilocycles/second (kcps) or megacycles/second (Mcps). In 1960 the scientific community adopted hertz as the unit of measure for frequency in honor of the German physicist, Heinrich Hertz, who first demonstrated radio wave transmission and reception in 1887.

age, we could *sample* it once every t_s seconds using an analog-to-digital converter and represent the sinewave as a sequence of discrete values. Plotting those individual values as dots would give us the discrete waveform in Figure 1–1(b). We say that Figure 1–1(b) is the "discrete-time" version of the continuous signal in Figure 1–1(a). The independent variable t in Eq. (1–1) and Figure 1–1(a) is continuous. The independent *index* variable n in Figure 1–1(b) is discrete and can have only integer values. That is, index n is used to identify the individual elements of the discrete sequence in Figure 1–1(b).

Do not be tempted to draw lines between the dots in Figure 1–1(b). For some reason, people (particularly those engineers experienced in working with continuous signals) want to connect the dots with straight lines, or the stairstep lines shown in Figure 1–1(c). Don't fall into this innocent-looking trap. Connecting the dots can mislead the beginner into forgetting that the $x(n)$ sequence is nothing more than a list of numbers. Remember, $x(n)$ is a discrete-time sequence of individual values, and each value in that sequence plots as a single dot. It's not that we're ignorant of what lies between the dots of $x(n)$; there *is* nothing between those dots.

We can reinforce this discrete-time sequence concept by listing those Figure 1–1(b) sampled values as follows:

$$\begin{aligned}
&x(0) = 0 && \text{(1st sequence value, index } n = 0)\\
&x(1) = 0.31 && \text{(2nd sequence value, index } n = 1)\\
&x(2) = 0.59 && \text{(3rd sequence value, index } n = 2)\\
&x(3) = 0.81 && \text{(4th sequence value, index } n = 3)\\
&\quad\ldots && \quad\ldots\\
& && \text{and so on,}
\end{aligned} \tag{1–2}$$

where n represents the time index integer sequence 0, 1, 2, 3, etc., and t_s is some constant time period. Those sample values can be represented collectively, and concisely, by the discrete-time expression

$$x(n) = \sin(2\pi f_o n t_s)\,. \tag{1–3}$$

(Here again, the $2\pi f_o n t_s$ term is an angle measured in radians.) Notice that the index n in Eq. (1–2) started with a value of 0, instead of 1. There's nothing sacred about this; the first value of n could just as well have been 1, but we start the index n at zero out of habit because doing so allows us to describe the sinewave starting at time zero. The variable $x(n)$ in Eq. (1–3) is read as "the sequence x of n." Equations (1–1) and (1–3) describe what are also referred to as *time-domain* signals because the independent variables, the continuous time t in Eq. (1–1), and the discrete-time nt_s values used in Eq. (1–3) are measures of time.

With this notion of a discrete-time signal in mind, let's say that a discrete system is a collection of hardware components, or software routines, that op-

erate on a discrete-time signal sequence. For example, a discrete system could be a process that gives us a discrete output sequence $y(0)$, $y(1)$, $y(2)$, etc., when a discrete input sequence of $x(0)$, $x(1)$, $x(2)$, etc., is applied to the system input as shown in Figure 1–2(a). Again, to keep the notation concise and still keep track of individual elements of the input and output sequences, an abbreviated notation is used as shown in Figure 1–2(b) where n represents the integer sequence 0, 1, 2, 3, etc. Thus, $x(n)$ and $y(n)$ are general variables that represent two separate sequences of numbers. Figure 1–2(b) allows us to describe a system's output with a simple expression such as

$$y(n) = 2x(n) - 1 \, . \tag{1–4}$$

Illustrating Eq. (1–4), if $x(n)$ is the five-element sequence: $x(0) = 1$, $x(1) = 3$, $x(2) = 5$, $x(3) = 7$, and $x(4) = 9$, then $y(n)$ is the five-element sequence $y(0) = 1$, $y(1) = 5$, $y(2) = 9$, $y(3) = 13$, and $y(4) = 17$.

The fundamental difference between the way time is represented in continuous and discrete systems leads to a very important difference in how we characterize frequency in continuous and discrete systems. To illustrate, let's reconsider the continuous sinewave in Figure 1–1(a). If it represented a voltage at the end of a cable, we could measure its frequency by applying it to an oscilloscope, a spectrum analyzer, or a frequency counter. We'd have a problem, however, if we were merely given the list of values from Eq. (1–2) and asked to determine the frequency of the waveform they represent. We'd graph those discrete values, and, sure enough, we'd recognize a single sinewave as in Figure 1–1(b). We can say that the sinewave repeats every 20 samples, but there's no way to determine the exact sinewave frequency from the discrete sequence values alone. You can probably see the point we're leading to here. If we knew the time between samples—the sample period t_s—we'd be able to determine the absolute frequency of the discrete sinewave.

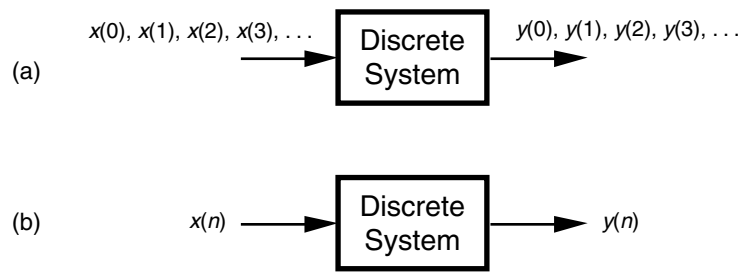


Figure 1–2 With an input applied, a discrete system provides an output: (a) the input and output are sequences of individual values; (b) input and output using the abbreviated notation of $x(n)$ and $y(n)$.

Given that the t_s sample period is, say, 0.05 milliseconds/sample, the period of the sinewave is

$$\text{sinewave period} = \frac{20 \text{ samples}}{\text{period}} \cdot \frac{0.05 \text{ milliseconds}}{\text{sample}} = 1 \text{ millisecond.} \quad (1\text{–}5)$$

Because the frequency of a sinewave is the reciprocal of its period, we now know that the sinewave's absolute frequency is 1/(1 ms), or 1 kHz. On the other hand, if we found that the sample period was, in fact, 2 milliseconds, the discrete samples in Figure 1–1(b) would represent a sinewave whose period is 40 milliseconds and whose frequency is 25 Hz. The point here is that, in discrete systems, absolute frequency determination in Hz is dependent on the sample frequency $f_s = 1/t_s$. We'll be reminded of this dependence throughout the rest of this book.

In digital signal processing, we often find it necessary to characterize the frequency content of discrete-time domain signals. When we do so, this frequency representation takes place in what's called the *frequency domain*. By way of example, let's say we have a discrete sinewave sequence $x_1(n)$ with an arbitrary frequency f_o Hz as shown on the left side of Figure 1–3(a). We can

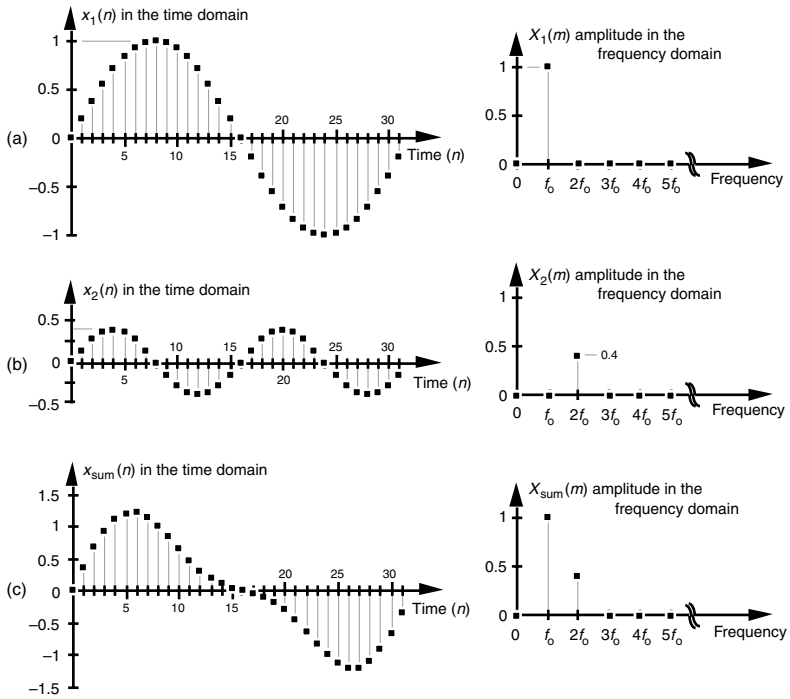


Figure 1–3 Time- and frequency-domain graphical representations: (a) sinewave of frequency f_o; (b) reduced amplitude sinewave of frequency $2f_o$; (c) sum of the two sinewaves.

also describe $x_1(n)$ as shown on the right side of Figure 1–3(a) by indicating that it has a frequency of 1, measured in units of f_o, and no other frequency content. Although we won't dwell on it just now, notice that the frequency-domain representations in Figure 1–3 are themselves discrete.

To illustrate our time- and frequency-domain representations further, Figure 1–3(b) shows another discrete sinewave $x_2(n)$, whose peak amplitude is 0.4, with a frequency of $2f_o$. The discrete sample values of $x_2(n)$ are expressed by the equation

$$x_2(n) = 0.4 \cdot \sin(2\pi 2 f_o n t_s) \,. \tag{1–6}$$

When the two sinewaves, $x_1(n)$ and $x_2(n)$, are added to produce a new waveform $x_{sum}(n)$, its time-domain equation is

$$x_{sum}(n) = x_1(n) + x_2(n) = \sin(2\pi f_o n t_s) + 0.4 \cdot \sin(2\pi 2 f_o n t_s) \,, \tag{1–7}$$

and its time- and frequency-domain representations are those given in Figure 1–3(c). We interpret the $X_{sum}(m)$ frequency-domain depiction, the *spectrum*, in Figure 1–3(c) to indicate that $X_{sum}(n)$ has a frequency component of f_o Hz and a reduced-amplitude frequency component of $2f_o$ Hz.

Notice three things in Figure 1–3. First, time sequences use lowercase variable names like the "x" in $x_1(n)$, and uppercase symbols for frequency-domain variables such as the "X" in $X_1(m)$. The term $X_1(m)$ is read as "the spectral sequence X sub one of m." Second, because the $X_1(m)$ frequency-domain representation of the $x_1(n)$ time sequence is itself a sequence (a list of numbers), we use the index "m" to keep track of individual elements in $X_1(m)$. We can list frequency-domain sequences just as we did with the time sequence in Eq. (1–2). For example $X_{sum}(m)$ is listed as

$X_{sum}(0) = 0$ (1st $X_{sum}(m)$ value, index $m = 0$)
$X_{sum}(1) = 1.0$ (2nd $X_{sum}(m)$ value, index $m = 1$)
$X_{sum}(2) = 0.4$ (3rd $X_{sum}(m)$ value, index $m = 2$)
$X_{sum}(3) = 0$ (4th $X_{sum}(m)$ value, index $m = 3$)
. . . . . .
and so on,

where the frequency index m is the integer sequence 0, 1, 2, 3, etc. Third, because the $x_1(n) + x_2(n)$ sinewaves have a phase shift of zero degrees relative to each other, we didn't really need to bother depicting this phase relationship in $X_{sum}(m)$ in Figure 1–3(c). In general, however, phase relationships in frequency-domain sequences are important, and we'll cover that subject in Chapters 3 and 5.

A key point to keep in mind here is that we now know three equivalent ways to describe a discrete-time waveform. Mathematically, we can use a time-domain equation like Eq. (1–6). We can also represent a time-domain waveform graphically as we did on the left side of Figure 1–3, and we can de-

pict its corresponding, discrete, frequency-domain equivalent as that on the right side of Figure 1–3.

As it turns out, the discrete-time domain signals we're concerned with are not only quantized in time; their amplitude values are also quantized. Because we represent all digital quantities with binary numbers, there's a limit to the resolution, or granularity, that we have in representing the values of discrete numbers. Although signal amplitude quantization can be an important consideration—we cover that particular topic in Chapter 12—we won't worry about it just now.

1.2 SIGNAL AMPLITUDE, MAGNITUDE, POWER

Let's define two important terms that we'll be using throughout this book: amplitude and magnitude. It's not surprising that, to the layman, these terms are typically used interchangeably. When we check our thesaurus, we find that they are synonymous.† In engineering, however, they mean two different things, and we must keep that difference clear in our discussions. The amplitude of a variable is the measure of how far, and in what direction, that variable differs from zero. Thus, signal amplitudes can be either positive or negative. The time-domain sequences in Figure 1–3 presented the sample value amplitudes of three different waveforms. Notice how some of the individual discrete amplitude values were positive and others were negative.

The magnitude of a variable, on the other hand, is the measure of how far, regardless of direction, its quantity differs from zero. So magnitudes are always positive values. Figure 1–4 illustrates how the magnitude of the $x_1(n)$

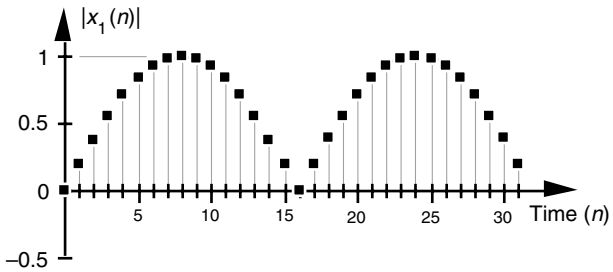


Figure 1–4 Magnitude samples, $|x_1(n)|$, of the time waveform in Figure 1–3(a).

† Of course, laymen are "other people." To the engineer, the brain surgeon is the layman. To the brain surgeon, the engineer is the layman.

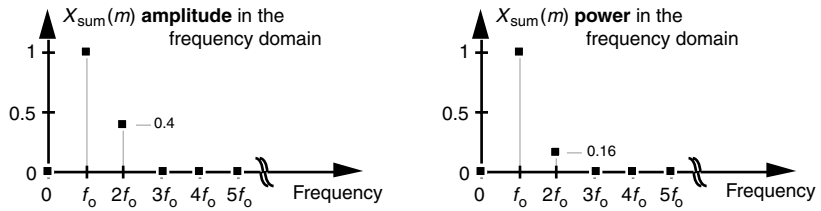


Figure 1–5 Frequency-domain amplitude and frequency-domain power of the $x_{sum}(n)$ time waveform in Figure 1–3(c).

time sequence in Figure 1–3(a) is equal to the amplitude, but with the sign always being positive for the magnitude. We use the modulus symbol (| |) to represent the magnitude of $x_1(n)$. Occasionally, in the literature of digital signal processing, we'll find the term *magnitude* referred to as the *absolute value*.

When we examine signals in the frequency domain, we'll often be interested in the power level of those signals. The power of a signal is proportional to its amplitude (or magnitude) squared. If we assume that the proportionality constant is one, we can express the power of a sequence in the time or frequency domains as

$$x_{pwr}(n) = x(n)^2 = |x(n)|^2, \tag{1–8}$$

or

$$X_{pwr}(m) = X(m)^2 = |X(m)|^2. \tag{1–8′}$$

Very often we'll want to know the difference in power levels of two signals in the frequency domain. Because of the squared nature of power, two signals with moderately different amplitudes will have a much larger difference in their relative powers. In Figure 1–3, for example, signal $x_1(n)$'s amplitude is 2.5 times the amplitude of signal $x_2(n)$, but its power level is 6.25 that of $x_2(n)$'s power level. This is illustrated in Figure 1–5 where both the amplitude and power of $X_{sum}(m)$ are shown.

Because of their squared nature, plots of power values often involve showing both very large and very small values on the same graph. To make these plots easier to generate and evaluate, practitioners usually employ the decibel scale as described in Appendix E.

1.3 SIGNAL PROCESSING OPERATIONAL SYMBOLS

We'll be using block diagrams to graphically depict the way digital signal-processing operations are implemented. Those block diagrams will comprise an assortment of fundamental processing symbols, the most common of which are illustrated and mathematically defined in Figure 1–6.

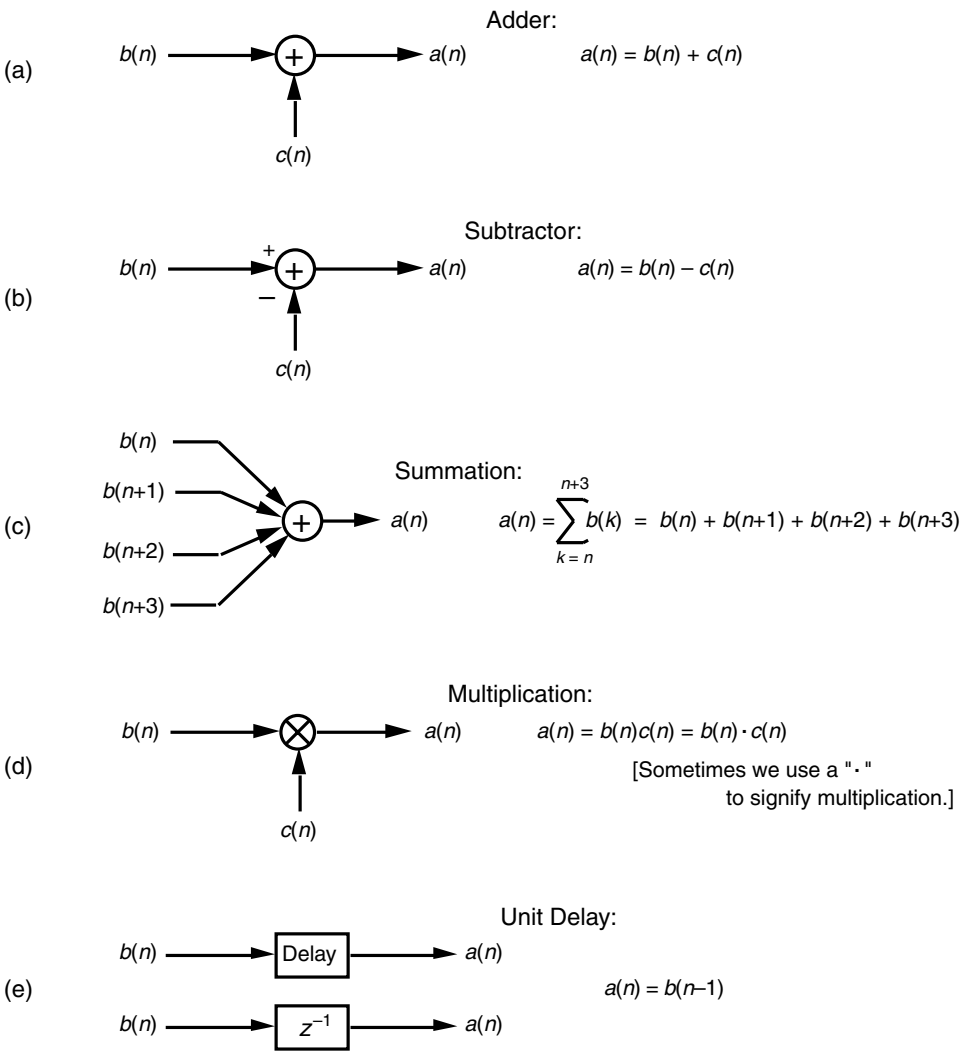


Figure 1–6 Terminology and symbols used in digital signal processing block diagrams.

Figure 1–6(a) shows the addition, element for element, of two discrete sequences to provide a new sequence. If our sequence index n begins at 0, we say that the first output sequence value is equal to the sum of the first element of the b sequence and the first element of the c sequence, or $a(0) = b(0) + c(0)$. Likewise, the second output sequence value is equal to the sum of the second element of the b sequence and the second element of the c sequence, or $a(1) = b(1) + c(1)$. Equation (1–7) is an example of adding two sequences. The

subtraction process in Figure 1–6(b) generates an output sequence that's the element-for-element difference of the two input sequences. There are times when we must calculate a sequence whose elements are the sum of more than two values. This operation, illustrated in Figure 1–6(c), is called summation and is very common in digital signal processing. Notice how the lower and upper limits of the summation index k in the expression in Figure 1–6(c) tell us exactly which elements of the b sequence to sum to obtain a given $a(n)$ value. Because we'll encounter summation operations so often, let's make sure we understand their notation. If we repeat the summation equation from Figure 1–6(c) here we have

$$a(n) = \sum_{k=n}^{n+3} b(k) \; . \tag{1–9}$$

This means that

$$\begin{array}{ll}
\text{when } n = 0, \text{ index } k \text{ goes from 0 to 3, so} & a(0) = b(0) + b(1) + b(2) + b(3) \\
\text{when } n = 1, \text{ index } k \text{ goes from 1 to 4, so} & a(1) = b(1) + b(2) + b(3) + b(4) \\
\text{when } n = 2, \text{ index } k \text{ goes from 2 to 5, so} & a(2) = b(2) + b(3) + b(4) + b(5) \\
\text{when } n = 3, \text{ index } k \text{ goes from 3 to 6, so} & a(3) = b(3) + b(4) + b(5) + b(6) \\
\quad \ldots & \quad \ldots \\
\text{and so on.} &
\end{array} \tag{1–10}$$

We'll begin using summation operations in earnest when we discuss digital filters in Chapter 5.

The multiplication of two sequences is symbolized in Figure 1–6(d). Multiplication generates an output sequence that's the element-for-element product of two input sequences: $a(0) = b(0)c(0)$, $a(1) = b(1)c(1)$, and so on. The last fundamental operation that we'll be using is called the *unit delay* in Figure 1–6(e). While we don't need to appreciate its importance at this point, we'll merely state that the unit delay symbol signifies an operation where the output sequence $a(n)$ is equal to a delayed version of the $b(n)$ sequence. For example, $a(5) = b(4)$, $a(6) = b(5)$, $a(7) = b(6)$, etc. As we'll see in Chapter 6, due to the mathematical techniques used to analyze digital filters, the unit delay is very often depicted using the term z^{-1}.

The symbols in Figure 1–6 remind us of two important aspects of digital signal processing. First, our processing operations are always performed on sequences of individual discrete values, and second, the elementary operations themselves are very simple. It's interesting that, regardless of how complicated they appear to be, the vast majority of digital signal processing algorithms can be performed using combinations of these simple operations. If we think of a digital signal processing algorithm as a recipe, then the symbols in Figure 1–6 are the ingredients.

1.4 INTRODUCTION TO DISCRETE LINEAR TIME-INVARIANT SYSTEMS

In keeping with tradition, we'll introduce the subject of linear time-invariant (LTI) systems at this early point in our text. Although an appreciation for LTI systems is not essential in studying the next three chapters of this book, when we begin exploring digital filters, we'll build on the strict definitions of linearity and time invariance. We need to recognize and understand the notions of linearity and time invariance not just because the vast majority of discrete systems used in practice are LTI systems, but because LTI systems are very accommodating when it comes to their analysis. That's good news for us because we can use straightforward methods to predict the performance of any digital signal processing scheme as long as it's linear and time invariant. Because linearity and time invariance are two important system characteristics having very special properties, we'll discuss them now.

1.5 DISCRETE LINEAR SYSTEMS

The term *linear* defines a special class of systems where the output is the superposition, or sum, of the individual outputs had the individual inputs been applied separately to the system. For example, we can say that the application of an input $x_1(n)$ to a system results in an output $y_1(n)$. We symbolize this situation with the following expression:

$$x_1(n) \xrightarrow{\text{results in}} y_1(n)\,. \tag{1–11}$$

Given a different input $x_2(n)$, the system has a $y_2(n)$ output as

$$x_2(n) \xrightarrow{\text{results in}} y_2(n)\ . \tag{1–12}$$

For the system to be linear, when its input is the sum $x_1(n) + x_2(n)$, its output must be the sum of the individual outputs so that

$$x_1(n)+x_2(n) \xrightarrow{\text{results in}} y_1(n)+y_2(n)\ . \tag{1–13}$$

One way to paraphrase expression (1–13) is to state that a linear system's output is the sum of the outputs of its parts. Also, part of this description of linearity is a proportionality characteristic. This means that if the inputs are scaled by constant factors c_1 and c_2 then the output sequence parts are also scaled by those factors as

$$c_1x_1(n)+c_2x_2(n) \xrightarrow{\text{results in}} c_1y_1(n)+c_2y_2(n)\ . \tag{1–14}$$

In the literature, this proportionality attribute of linear systems in expression (1–14) is sometimes called the *homogeneity property*. With these thoughts in mind, then, let's demonstrate the concept of system linearity.

1.5.1 Example of a Linear System

To illustrate system linearity, let's say we have the discrete system shown in Figure 1–7(a) whose output is defined as

$$y(n) = \frac{-x(n)}{2}\ , \tag{1–15}$$

that is, the output sequence is equal to the negative of the input sequence with the amplitude reduced by a factor of two. If we apply an $x_1(n)$ input sequence representing a 1-Hz sinewave sampled at a rate of 32 samples per cycle, we'll have a $y_1(n)$ output as shown in the center of Figure 1–7(b). The frequency-domain spectral amplitude of the $y_1(n)$ output is the plot on the

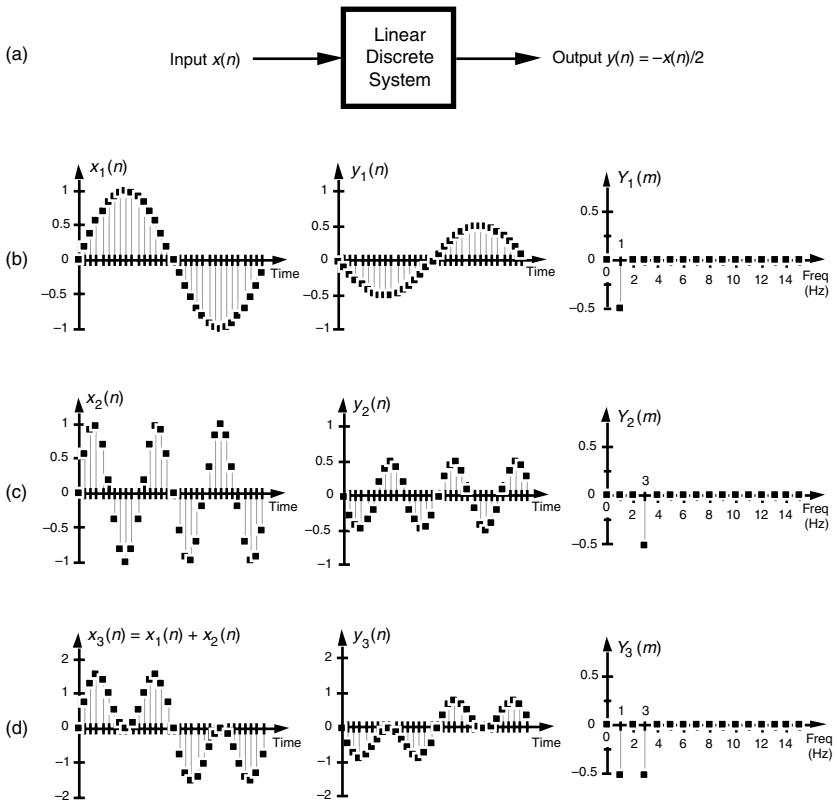


Figure 1–7 Linear system input-to-output relationships: (a) system block diagram where $y(n) = -x(n)/2$; (b) system input and output with a 1-Hz sinewave applied; (c) with a 3-Hz sinewave applied; (d) with the sum of 1-Hz and 3-Hz sinewaves applied.

right side of Figure 1–7(b), indicating that the output comprises a single tone of peak amplitude equal to –0.5 whose frequency is 1 Hz. Next, applying an $x_2(n)$ input sequence representing a 3-Hz sinewave, the system provides a $y_2(n)$ output sequence, as shown in the center of Figure 1–7(c). The spectrum of the $y_2(n)$ output, $Y_2(m)$, confirming a single 3-Hz sinewave output is shown on the right side of Figure 1–7(c). Finally—here's where the linearity comes in—if we apply an $x_3(n)$ input sequence that's the sum of a 1-Hz sinewave and a 3-Hz sinewave, the $y_3(n)$ output is as shown in the center of Figure 1–7(d). Notice how $y_3(n)$ is the sample-for-sample sum of $y_1(n)$ and $y_2(n)$. Figure 1–7(d) also shows that the output spectrum $Y_3(m)$ is the sum of $Y_1(m)$ and $Y_2(m)$. That's linearity.

1.5.2 Example of a Nonlinear System

It's easy to demonstrate how a nonlinear system yields an output that is not equal to the sum of $y_1(n)$ and $y_2(n)$ when its input is $x_1(n) + x_2(n)$. A simple example of a nonlinear discrete system is that in Figure 1–8(a) where the output is the square of the input described by

$$y(n) = [x(n)]^2 . \tag{1–16}$$

We'll use a well known trigonometric identity and a little algebra to predict the output of this nonlinear system when the input comprises simple sinewaves. Following the form of Eq. (1–3), let's describe a sinusoidal sequence, whose frequency $f_o = 1$ Hz, by

$$x_1(n) = \sin(2\pi f_o n t_s) = \sin(2\pi \cdot 1 \cdot n t_s) . \tag{1–17}$$

Equation (1–17) describes the $x_1(n)$ sequence on the left side of Figure 1–8(b). Given this $x_1(n)$ input sequence, the $y_1(n)$ output of the nonlinear system is the square of a 1-Hz sinewave, or

$$y_1(n) = [x_1(n)]^2 = \sin(2\pi \cdot 1 \cdot n t_s) \cdot \sin(2\pi \cdot 1 \cdot n t_s) . \tag{1–18}$$

We can simplify our expression for $y_1(n)$ in Eq. (1–18) by using the following trigonometric identity:

$$\sin(\alpha) \cdot \sin(\beta) = \frac{\cos(\alpha - \beta)}{2} - \frac{\cos(\alpha + \beta)}{2} . \tag{1–19}$$

Using Eq. (1–19), we can express $y_1(n)$ as

$$\begin{aligned} y_1(n) &= \frac{\cos(2\pi \cdot 1 \cdot n t_s - 2\pi \cdot 1 \cdot n t_s)}{2} - \frac{\cos(2\pi \cdot 1 \cdot n t_s + 2\pi \cdot 1 \cdot n t_s)}{2} \\ &= \frac{\cos(0)}{2} - \frac{\cos(4\pi \cdot 1 \cdot n t_s)}{2} = \frac{1}{2} - \frac{\cos(2\pi \cdot 2 \cdot n t_s)}{2} , \end{aligned} \tag{1–20}$$

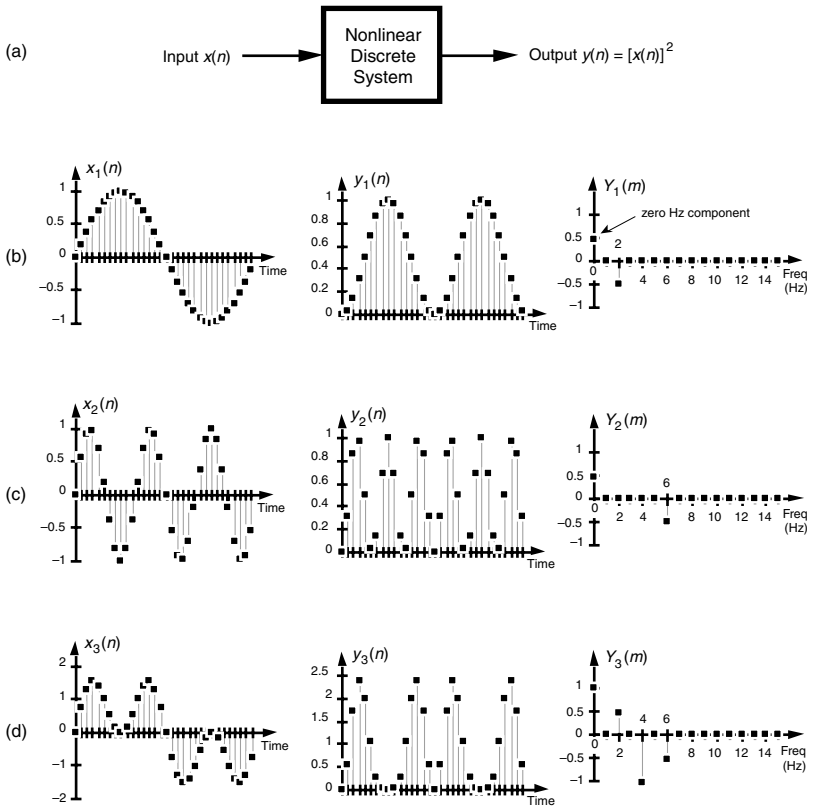


Figure 1–8 Nonlinear system input-to-output relationships: (a) system block diagram where $y(n) = (x(n))^2$; (b) system input and output with a 1-Hz sinewave applied; (c) with a 3-Hz sinewave applied; (d) with the sum of 1-Hz and 3-Hz sinewaves applied.

which is shown as the all positive sequence in the center of Figure 1–8(b). Because Eq. (1–19) results in a frequency sum ($\alpha + \beta$) and frequency difference ($\alpha - \beta$) effect when multiplying two sinusoids, the $y_1(n)$ output sequence will be a cosine wave of 2 Hz and a peak amplitude of –0.5, added to a constant value of 1/2. The constant value of 1/2 in Eq. (1–20) is interpreted as a zero Hz frequency component, as shown in the $Y_1(m)$ spectrum in Figure 1–8(b). We could go through the same algebraic exercise to determine that, when a 3-Hz sinewave $x_2(n)$ sequence is applied to this nonlinear system, the output $y_2(n)$ would contain a zero Hz component and a 6 Hz component, as shown in Figure 1–8(c).

System nonlinearity is evident if we apply an $x_3(n)$ sequence comprising the sum of a 1-Hz and a 3-Hz sinewave as shown in Figure 1–8(d). We can

predict the frequency content of the $y_3(n)$ output sequence by using the algebraic relationship

$$(a+b)^2 = a^2+2ab+b^2\,, \tag{1–21}$$

where a and b represent the 1-Hz and 3-Hz sinewaves, respectively. From Eq. (1–19), the a^2 term in Eq. (1–21) generates the zero-Hz and 2-Hz output sinusoids in Figure 1–8(b). Likewise, the b^2 term produces in $y_3(n)$ another zero-Hz and the 6-Hz sinusoid in Figure 1–8(c). However, the $2ab$ term yields additional 2-Hz and 4-Hz sinusoids in $y_3(n)$. We can show this algebraically by using Eq. (1–19) and expressing the $2ab$ term in Eq. (1–21) as

$$\begin{aligned} 2ab &= 2\cdot\sin(2\pi\cdot 1\cdot nt_s)\cdot\sin(2\pi\cdot 3\cdot nt_s) \\ &= \frac{2\cos(2\pi\cdot 1\cdot nt_s - 2\pi\cdot 3\cdot nt_s)}{2} - \frac{2\cos(2\pi\cdot 1\cdot nt_s + 2\pi\cdot 3\cdot nt_s)}{2} \\ &= \cos(2\pi\cdot 2\cdot nt_s) - \cos(2\pi\cdot 4\cdot nt_s)\ .^{\dagger} \end{aligned} \tag{1–22}$$

Equation (1–22) tells us that two additional sinusoidal components will be present in $y_3(n)$ because of the system's nonlinearity, a 2-Hz cosine wave whose amplitude is +1 and a 4-Hz cosine wave having an amplitude of –1. These spectral components are illustrated in $Y_3(m)$ on the right side of Figure 1–8(d).

Notice that, when the sum of the two sinewaves is applied to the nonlinear system, the output contained sinusoids, Eq. (1–22), that were not present in either of the outputs when the individual sinewaves alone were applied. Those extra sinusoids were generated by an interaction of the two input sinusoids due to the squaring operation. That's nonlinearity; expression (1–13) was not satisfied. (Electrical engineers recognize this effect of internally generated sinusoids as *intermodulation distortion*.) Although nonlinear systems are usually difficult to analyze, they are occasionally used in practice. References [2], [3], and [4], for example, describe their application in nonlinear digital filters. Again, expressions (1–13) and (1–14) state that a linear system's output resulting from a sum of individual inputs, is the superposition (sum) of the individual outputs. They also stipulate that the output sequence $y_1(n)$ depends only on $x_1(n)$ combined with the system characteristics, and not on the other input $x_2(n)$, i.e., there's no interaction between inputs $x_1(n)$ and $x_2(n)$ at the output of a linear system.

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† The first term in Eq. (1–22) is $\cos(2\pi\cdot nt_s - 6\pi\cdot nt_s) = \cos(-4\pi\cdot nt_s) = \cos(-2\pi\cdot 2\cdot nt_s)$. However, because the cosine function is even, $\cos(-\alpha) = \cos(\alpha)$, we can express that first term as $\cos(2\pi\cdot 2\cdot nt_s)$.

1.6 TIME-INVARIANT SYSTEMS

A time-invariant system is one where a time delay (or shift) in the input sequence causes a equivalent time delay in the system's output sequence. Keeping in mind that n is just an indexing variable we use to keep track of our input and output samples, let's say a system provides an output $y(n)$ given an input of $x(n)$, or

$$x(n) \xrightarrow{\text{results in}} y(n) \ . \tag{1–23}$$

For a system to be time invariant, with a shifted version of the original $x(n)$ input applied, $x'(n)$, the following applies:

$$x'(n) = x(n+k) \xrightarrow{\text{results in}} y'(n) = y(n+k) \ , \tag{1–24}$$

where k is some integer representing k sample period time delays. For a system to be time invariant, expression (1–24) must hold true for any integer value of k and any input sequence.

1.6.1 Example of a Time-Invariant System

Let's look at a simple example of time invariance illustrated in Figure 1–9. Assume that our initial $x(n)$ input is a unity-amplitude 1-Hz sinewave sequence with a $y(n)$ output, as shown in Figure 1–9(b). Consider a different input sequence $x'(n)$, where

$$x'(n) = x(n+4) \ . \tag{1–25}$$

Equation (1–25) tells us that the input sequence $x'(n)$ is equal to sequence $x(n)$ shifted four samples to the left, that is, $x'(0) = x(4)$, $x'(1) = x(5)$, $x'(2) = x(6)$, and so on, as shown on the left of Figure 1–9(c). The discrete system is time invariant because the $y'(n)$ output sequence is equal to the $y(n)$ sequence shifted to the left by four samples, or $y'(n) = y(n+4)$. We can see that $y'(0) = y(4)$, $y'(1) = y(5)$, $y'(2) = y(6)$, and so on, as shown in Figure 1–9(c). For time-invariant systems, the y time shift is equal to the x time shift.

Some authors succumb to the urge to define a time-invariant system as one whose parameters do not change with time. That definition is incomplete and can get us in trouble if we're not careful. We'll just stick with the formal definition that a time-invariant system is one where a time shift in an input sequence results in an equal time shift in the output sequence. By the way, time-invariant systems in the literature are often called *shift-invariant* systems.[†]

[†] An example of a discrete process that's not time-invariant is the downsampling, or decimation, process described in Chapter 10.

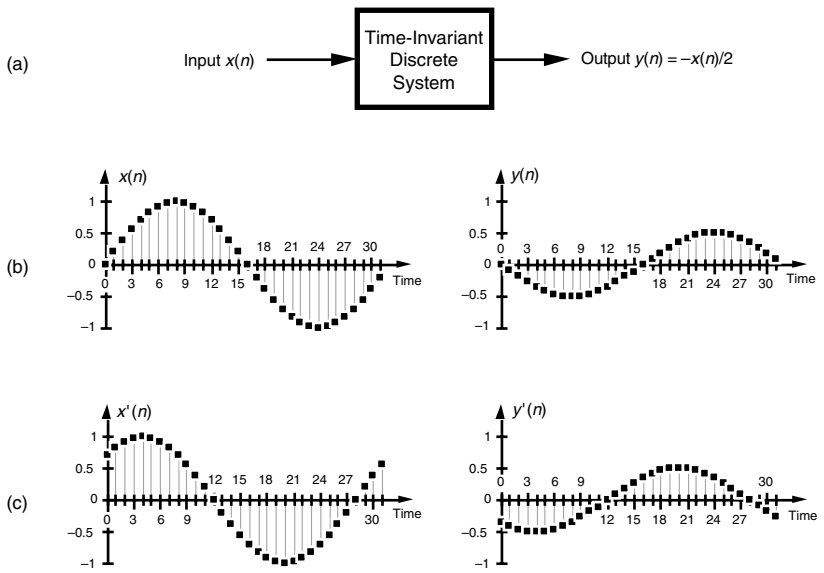

Figure 1–9 Time-invariant system input-to-output relationships: (a) system block diagram where $y(n) = -x(n)/2$; (b) system input and output with a 1-Hz sinewave applied; (c) system input and output when a 1-Hz sinewave, delayed by four samples, is applied. When $x'(n) = x(n+4)$, then, $y'(n) = y(n+4)$.

1.7 THE COMMUTATIVE PROPERTY OF LINEAR TIME-INVARIANT SYSTEMS

Although we don't substantiate this fact until we reach Section 6.8, it's not too early to realize that LTI systems have a useful commutative property by which their sequential order can be rearranged with no change in their final output. This situation is shown in Figure 1–10 where two different LTI

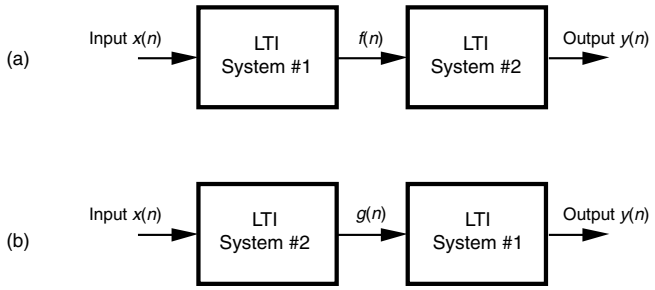

Figure 1–10 Linear time-invariant (LTI) systems in series: (a) block diagram of two LTI systems; (b) swapping the order of the two systems does not change the resultant output $y(n)$.

systems are configured in series. Swapping the order of two cascaded systems does not alter the final output. Although the intermediate data sequences $f(n)$ and $g(n)$ will usually not be equal, the two pairs of LTI systems will have identical $y(n)$ output sequences. This commutative characteristic comes in handy for designers of digital filters, as we'll see in Chapters 5 and 6.

1.8 ANALYZING LINEAR TIME-INVARIANT SYSTEMS

As previously stated, LTI systems can be analyzed to predict their performance. Specifically, if we know the *unit impulse response* of an LTI system, we can calculate everything there is to know about the system; that is, the system's unit impulse response completely characterizes the system. By unit impulse response, we mean the system's time-domain output sequence when the input is a single unity-valued sample (unit impulse) preceded and followed by zero-valued samples as shown in Figure 1–11(b).

Knowing the (unit) impulse response of an LTI system, we can determine the system's output sequence for any input sequence because the output is equal to the *convolution* of the input sequence and the system's impulse response. Moreover, given an LTI system's time-domain impulse response, we can find the system's *frequency response* by taking the Fourier transform in the form of a *discrete Fourier transform* of that impulse response[5].

Don't be alarmed if you're not exactly sure what is meant by convolution, frequency response, or the discrete Fourier transform. We'll introduce these subjects and define them slowly and carefully as we need them in later chapters. The point to keep in mind here is that LTI systems can be designed and analyzed using a number of straightforward and powerful analysis tech-

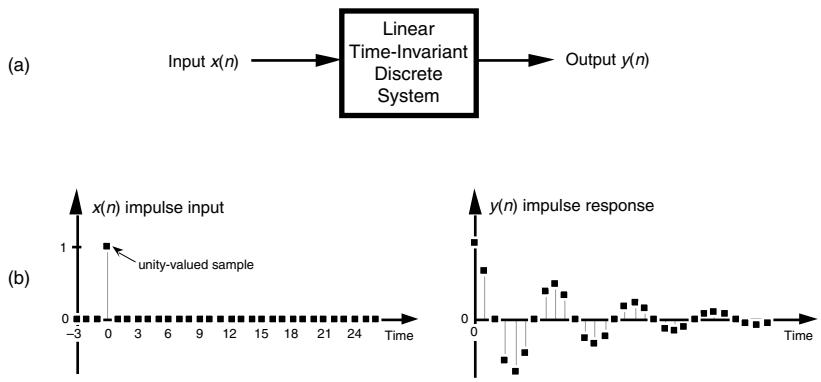

Figure 1–11 LTI system unit impulse response sequences: (a) system block diagram; (b) impulse input sequence $x(n)$ and impulse response output sequence $y(n)$.

niques. These techniques will become tools that we'll add to our signal processing toolboxes as we journey through the subject of digital signal processing.

REFERENCES

[1] Karplus, W. J., and Soroka, W. W. *Analog Methods*, Second Edition, McGraw-Hill, New York, 1959, p. 117.

[2] Mikami, N., Kobayashi, M., and Yokoyama, Y. "A New DSP-Oriented Algorithm for Calculation of the Square Root Using a Nonlinear Digital Filter," *IEEE Trans. on Signal Processing*, Vol. 40, No. 7, July 1992.

[3] Heinen, P., and Neuvo, Y. "FIR-Median Hybrid Filters," *IEEE Trans. on Acoust. Speech, and Signal Processing*, Vol. ASSP-35, No. 6, June 1987.

[4] Oppenheim, A., Schafer, R., and Stockham, T. "Nonlinear Filtering of Multiplied and Convolved Signals," *Proc. IEEE*, Vol. 56, August 1968.

[5] Pickerd, John. "Impulse-Response Testing Lets a Single Test Do the Work of Thousands," *EDN*, April 27, 1995.